J. Math. Res. & Exposition, 1986, No. 2

Hom Functors from A Regular Category*

Yu Yong-Xi (于永溪)

(Suzhou University, Suzhou)

This paper aims to show the exactness of Hom functors with domain a regular category and codomain the category set.

1° **In** [1], we put forward a problem:

To what degree and to which categories with ferminal objects can HAA (homological algebra in abelian categories) be naturally extended? By "naturally" we mean that the generalized theory degenerate into the theory of HAA when the terminal objects are nulls.

This problem is called the natural homological algebra problem or NHA for short.

In [1], we completed the second step to establish NHA, i.e., we knew the generalized exact sequences corresponding to the quasikernels (see [2]).

If F is a terminal object and if the diagram $\begin{tikzcd} {} \arrow[r, "s"] \arrow[d, "u"'] & F \arrow[d, "h"] \\ {} \arrow[r, "f"'] & {} \end{tikzcd}$ is a pullback, then u is called a quasikernel of f and h is called a lateral of u. $QK^F(f)$ denotes all the quasikernels of f.

As is well-known, Hom functors play important roles in HAA, undoubtedly, it is important for NHA to discuss the exactness of Hom functors from a regular category.

In order to read smoothly, we collect all the relevant facts in [1] as follows:

Axiom(P). If there is a commutative diagram $\begin{tikzcd} {} \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "f"] & {} \arrow[d, "g"] \\ {} \arrow[r] & {} \end{tikzcd}$ with g a monic, then there is a pullback $\begin{tikzcd} {} \arrow[r, "b"] \arrow[d, "a"'] & {} \arrow[d, "g"] \\ {} \arrow[r, "f"'] & {} \end{tikzcd}$.

Axiom (U). If $f: A \twoheadrightarrow B$ is a regular epi and a_i is a subobject of B, $i \in T$, such that $\tilde{\bigcup\limits_T} a_i = \langle 1_B \rangle$, then $\tilde{\bigcup\limits_T} f^{-1}(a_i) = \langle 1_A \rangle$.

Axiom (CQN). Every epi is a terminal coquasikernel with any lateral.

An epi is called a (CQN)-morphism, if it is a terminal coquasikernel with any lateral. That is, an epi $f: A \twoheadrightarrow B$ is called a (CQN)-morphism, if for any morphism $h \in (F, B)$ there is a pushout $\begin{tikzcd} {} \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "f"] & F \arrow[d] \\ {} \arrow[r] & B \end{tikzcd}$.

If $a_i \leqslant a$, and if a_i is carried into h by 1, a is carried into h by 1

* Received June 21, 1984.

also, then we call a the weak-union of $\{a_i\}$. we write $\tilde{\bigcup} a_i$ for it .

Lemma 1.2. For any category with a terminal object, a quasikernel is an h-lateral quasikernel of its h-lateral coquasikernel and an h-lateral coquasikernel is an h-lateral coquasikernel of its h-lateral quasikernel.

Lemma 1.3. For a (P)-category, $\bigcup a_i = \tilde{\bigcup} a_i$ when $\tilde{\bigcup} a_i$ exists.

Lemma 1.5. For a category, if $\tilde{\bigcup} a_i$ exists and $(\tilde{\bigcup} a_i) t_i = a_i$, then $\tilde{\bigcup} t_i = \langle 1 \rangle$.

Proposition 2.2. For any category, if $\{u_i\}_{i\in T}$ is a family of subobjects u_i's of f, and if $\bigcup_T u_i$, $f(\bigcup_T u_i)$ and $f(u_i)$ exist, then $\tilde{\bigcup}_T f(u_i) = f(\bigcup_T u_i)$.

Proposition 2.7. If $\mathscr{A}$ is a (P)-weak regular category, then $\mathscr{A}$ satisfies Axiom (U) $\Longleftrightarrow$ for every regular epi f , $\bigcup f^{-1}(h_i) = (f^{-1}(\bigcup h_i)$ when $\bigcup h_i$ exists.

Definition 3.1. A sequence $A \xrightarrow{g} B \xrightarrow{f} C$ is called to be I-exact at B , if g has an image and there is a family $\{a_i\}_{i\in T}$ of subobjects a_i's of A such that $\bigcup_T a_i = \langle 1 \rangle$ and $\{g(a_i)\}_T = QK^F(f)$.

Definition 3.2. A sequence $A \xrightarrow{g} B \xrightarrow{f} C$ is called to be I′-exact at B, if g has an image and for each $\langle u_i \rangle \in QK^F(f)$ exists the image $g^{-1}(u_i) = \langle a_i \rangle$ such that $\bigcup a_i = \langle 1_A \rangle$ and $\{g(a_i)\}_i = QK^F(f)$.

Definition 3.3. A sequence $A \xrightarrow{g} B \xrightarrow{f} C$ is called to be II-exact at B ,

if $\bigcup\limits_{u_i \in QK^F(f)} u_i$ exists and $\bigcup\limits_{n_i \in QK^F(f)} u_i = \mathrm{Im}(g)$.

Definition 3.4. Let $\langle u \rangle \in QK^F(f)$, a sequence $A \xrightarrow{g} B \xrightarrow{f} C$ is called to be exact by the component $\langle u \rangle$ at B, if $\mathrm{Im}(g) = \langle u \rangle$.

Proposition 3.8. For any category, an I-exact sequence must be II-exact.

Theorem 3.9. For any (P)-category, if g is monic, then I-exactness, I′-exactness, and II-exactness are equivalent.

2° **In this paper** F always denotes a terminal object. The regular categories are always understood in M. Barr′s sense (see [3]). The symbols in this paper are the same as those in [1], The categories in this paper always have terminal objects.

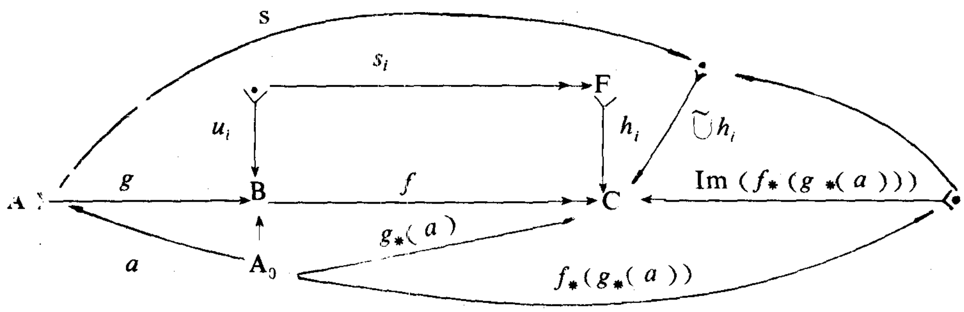
We shall use the following symbols:

Given a sequence $A \xrightarrow{g} B \xrightarrow{f} C$, we appoint that $(A_0, A) \xrightarrow{g_*} (A_0, B) \xrightarrow{f_*} (A_0, C)$ is the sequence induced by $\mathrm{Hom}(A_0, -)$, $(C, B_0) \xrightarrow{f^*} (B, B_0) \xrightarrow{g^*} (A, B_0)$ is the sequence induced by $\mathrm{Hom}(-, B_0)$.

Suppose $\langle u_i \rangle \in QK^F(f)$, h_i is a lateral of u_i, $i \in T$, $b_j \in (F, B_0)$. Let $QK(f_*) = \{v \mid v \in (A_0, B)$ and $\mathrm{Im}(fv) \leqslant \tilde{\bigcup}_T h_i\}$ and $QK(g^*) = \{w \mid w \in (B, B_0)$ and $\mathrm{Im}(wg) = \tilde{\bigcup}' b_j\}$, where $\tilde{\bigcup}'$ denotes a partial weak union.

Proposition 1. For a (P)-regular category, if $A \rightarrowtail^{g} B \twoheadrightarrow^{f} C$ is generalized exact at B, then for any object A_0, $g_*((A_0, A)) \subset \widetilde{QK(f_*)}$.

Proof. We hope to complete the proof by discussing the following diagram:



By proposition 3.8, we can suppose the sequence is II-exact at B, so we have $\langle g \rangle = \widetilde{\bigcup} u_i$. At first, we hope to prove $\widetilde{\bigcup} h_i$ exists. In fact, because h_i is a lateral of u_i, Axiom (EX 1) (see[3]) tells us that s_i in the pullback $(f, h_i; u_i, s_i)$ is regular epi, so that $f(u_i) = \langle h_i \rangle$. Since a regular category always has images, $f(\widetilde{\bigcup} u_i)$ exists. From Lemma 1.3 and proposition 2.2, we know $\bigcup h_i$ exists and $f(g) = \bigcup h_i$. Hence there is a regular epi s such that $fg = (\bigcup h_i)s$. Let $a \in (A_0, A)$ be any morphism, we have $ga = g_*(a) \in (A_0, B)$. In addition, $f_*(g_*(a)) = f \cdot (g_*(a)) = fga = (\bigcup h_i)sa$. Since $(\bigcup h_i)$ is monic, the definition of images shows $\mathrm{Im}(f_*(g_*(a))) \leqslant \bigcup h_i$, so that $ga = g_*(a) \in \widetilde{QK(f_*)}$. Therefore, , $g_*((A_0, A)) \subset \widetilde{QK(f_*)}$, q. e. d.

When the sequence is exact by a component$\langle u_i \rangle$, the proof is obvious. But in this case we have $g_*((A_0, A)) \subset \{v \mid v \in (A_0, B)$ and $\mathrm{Im}(fv) \leqslant h_i\}$.

Proposition 2. For a (P) (U)-regular category, if $A \xrightarrow{g} B \xrightarrow{f} C$ is I-, I′-, or II-exact at B, then $g_*((A_0, A)) = \widetilde{QK(f_*)}$; if the sequence is exact by a component u_i at B, then $g_*((A_0, A)) = \{v \mid v \in (A_0, B$ and $\mathrm{Im}(fv) \leqslant h_i\}$.

proof. From Proposition 3.8, we can suppose the sequence is II-exact. To refer the proof of Proposition 1 we know $f(g) = \bigcup h_i$. Since f is regular epi, Axiom (EX1) shows there is a pullback $(f, \bigcup h_i; s, d)$ and $\langle s \rangle = f^{-1}(\bigcup h_i) = \bigcup f^{-1}(h_i) = \bigcup u_i = \langle g \rangle$. Now let $v \in \widetilde{QK(f)}_*$, that is, $v \in (A_0, B)$ and $\mathrm{Im}(fv) \leqslant \bigcup h_i$, by the definition of images there is a morphism l such that $fv = (\bigcup h_i)l$, so since $(f, \bigcup h_i; s, d)$ is a pullback, there is a morphism t such that $st = v$. Because $s \cong g$ as above, there is a morphism $t' \in (A_0, A)$ such that $gt' = v$, so that $QK(f_*) \subset g_*((A_0, A))$.

When the sequence is exact by a component the proof is very easy, q.e.d.

Proposition 3. For a (P)-regular category, if $A \xrightarrow{g} B \xrightarrow{f} C$ is generalized exact at B, then for any object B_0, $f^*((C, B_0)) = \mathrm{Im}(f^*) \subset \widetilde{QK(g^*)}$. In particular, if the sequence is exact by a component u_i, then $\mathrm{Im}(f^*) \subset \{w \mid w \in (B, B_0)$ and

$\operatorname{Im}(wg)=b$, where b is an elenment of $(F, B_0)\}$.

Proof. Suppose the sequence is II-exact at B, the proof can be completed by the following discussion:

Let $u_i \in QK^F(f)$, then we have a pullback $(f, h_j; u_j, s_j)$. Axiom (EX 1) shows s_j is regular epi. Since $\langle g \rangle = \bigcup u_i$, there is a unique a_j such that $ga_j = u_j$. New let $s \in (C, B_0)$, then $f^*(s) = sf \in (B, B_0)$ and $(g^*(f^*(s)))a_j = ((sf)g)a_i = s\cdot(fu_j) = s\cdot(h_j s_j) = (sh_j)s_j = b_j s_j$, where $b_j = sh_j \in (F, B_0)$. Since s_j is regular epi and b_j is monic, the image $(sfg)(a_j)$ is equal to $\langle b_j \rangle$. On the other hand, Lemma 1.5 tells us $\widetilde{\bigcup} a_i = \langle 1 \rangle$. In addition, Lemma 1.3 shows $\bigcup a_i = \widetilde{\bigcup} a_i = \langle 1 \rangle$. By Proposition 2.2, we have $\widetilde{\bigcup}(sfg)(a_i) = (sfg)(\bigcup a_i) = (sfg)(1) = \operatorname{Im}(sfg)$. since we have proved $(sfg)(a_j) = \langle b_j \rangle$, $\widetilde{\bigcup}(sfg)(a_i) = \widetilde{\bigcup}' b_i$, hence $\operatorname{Im}((f^*(s))g) = \widetilde{\bigcup}' b_i$, so that $\operatorname{Im}(f^*) \subset \widehat{QK(g^*)}$, q. e. d.

Proposition 4. For a (CQN)-category, if $A \xrightarrow{g} B \xrightarrow{f} C$ is generalized exact at B, then when $(F, B_0) = \{b_0\}$, $QK(g^*) = \operatorname{Im}(f^*)$ holds.

Proof. Let $u_j \in QK^F(f)$, we have a pullback $(f, h_i; u_j, s_j)$. Suppose the sequence is II-exact at B, then we have $\langle g \rangle = \widetilde{\bigcup} u_i$, Hence there is an unique a_j such that $ga_j = u_j$. Now let $s \in \widehat{QK(g^*)}$, so $s \in (B, B_0)$ and $\operatorname{Im}(sg) = \widetilde{\bigcup}' b_j = \langle b_0 \rangle$. Hence there is a morphism a such that $sg = b_0 a$, so $s(ga_j) = b_0(aa_j) = b_0 s_j$. On the other hand, since f is regular epi and $ga_j (= u_j)$ is ist h_j-lateral quasikernel, Lemma 1.2 shows that f is an h_j-lateral coquasikernel of u_j, that is, $(f, h_j; u_j, s_j)$ is a pushout. In addition, $su_j = (sg)a_j = b_0(aa_j) = b_0 s_j$, so there is a morphism $t \in (C, B_0)$ such that $tf = s$ or $s = f^*(t)$, so that $\widehat{QK(g^*)} \subset f^*((C, B_0))$. The discussion as above is a proof for the case that the sequence is exact by a component, q. e. d.

References

[1] Yu Yong-Xi, On generalized exact sequences, to appear.

[2] Yu Yong-xi, The quasikernels in an n-preadditive category, J. Math. Res. and Exposition, first issue (1981), 7—15.

[3] M. Barr, Exact categories, Lecture Notes in Mathematics 236 (Springer, Berlin, 1971), 1—120.