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On the Approximation of Combination of Linear Operators*

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Let $f(x)\in C_{2\pi}$. For Vallée-Poussin integrals

$$V_n(f,x)=\frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+t)\cos^{2n}\frac{t}{2}dt,$$

Z.Ditzian and G.Freud[1] considered the approximation of their combination writ ing

$$V_{n,1}(f,x)=2V_{2n-1}(f,x)-V_{n-1}(f,x),$$

$$V_{n,2}(f,x)=\frac{8}{3}V_{4n-1}(f,x)-2V_{2n-1}(f,x)+\frac{1}{3}V_{n-1}(f,x),$$

they proved that

$$V_{n,1}(f,x)-f(x)=O(\omega_4(f,\frac{1}{\sqrt{n}})),$$

$$V_{n,2}(f,x)-f(x)=O(\omega_6(f,\frac{1}{\sqrt{n}}))$$

In this paper, using the asymptotic expansions of linear operators with many terms, we generalize the above result to the case of eombination of m terms, where m is an arbtirary positive integer.

Let $q_1, q_2, \cdots, q_m$ be positive integers and $1<q_1<q_2<\cdots<q_m$. Suppose that numbers $A_0^{(m)}+A_1^{(m)}+\cdots+A_m^{(m)}$ satisfy the bollowing nonhomogeneous linear equation system:

$$\begin{aligned}
&A_0^{(m)}+A_1^{(m)}+\cdots+A_m^{(m)}=1,\\
&A_0^{(m)}+\frac{1}{q_1}A_1^{(m)}+\cdots+\frac{1}{q_m}A_m^{(m)}=0,\\
&A_0^{(m)}+\frac{1}{q_1^2}A_1^{(m)}+\cdots+\frac{1}{q_m^2}A_m^{(m)}=0,\\
&A_0^{(m)}+\frac{1}{q_1^m}A_1^{(m)}+\cdots+\frac{1}{q_m^m}A_m^{(m)}=0
\end{aligned}\qquad (1)$$

Since $1, x, x^2, \cdots, x^m$ are linear independent, we have

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$$\begin{vmatrix} 1 & 1 \cdots 1 \\ 1 & \frac{1}{q_1} \cdots \frac{1}{q_m} \\ 1 & \frac{1}{q_1^2} \cdots \frac{1}{q_m^2} \\ 1 & \frac{1}{q_1^m} \cdots \frac{1}{q_m^m} \end{vmatrix} \neq 0$$

Therefore the non-trivial solution $\{A_0^{(m)}, A_1^{(m)}, \cdots, A_m^{(m)}\}$ exists and is unique. In this case, we can write

$$V^q_{n,m}(f,x) = A_m^{(m)} V_{q_m n-1}(f,x) + \cdots + A_1^{(m)} V_{q_1 n-1}(f,x) + A_0^{(m)} V_{n-1}(f,x).$$

We shall prove the following

Theorem 2 For $f(x) \in C_{2\pi}$, we have

$$V^q_{n,m}(f,x) - f(x) = O(\omega_{2m+2}(f, \frac{1}{\sqrt{n}})).$$

It is easy to verify that Z.Ditzian and G.Freud's result is the special case of the above theorem whem $m = 1$ or 2, $q_1 = 2$, $q_2 = 4$.

For $m = 2$, $q_1 = 2$, $u_2 = 3$, we can obtain $A_0^{(2)} = \frac{1}{2}$, $A_1^{(2)} = -4$, $A_2^{(2)} = \frac{9}{2}$ by solving the equation system (1). Write

$$\overline{V}_{n,2}(f,x) = \frac{9}{2} V_{3n-1}(f,x) - 4 V_{2n-1}(f,x) + \frac{1}{2} V_{n-1}(f,x).$$

According to theorem 2, we have

Corollary I Suppose that $f(x) \in C_{2\pi}$. Then

$$\overline{V}_{n,2}(f,x) - f(x) = O(\omega_6(f, \frac{1}{\sqrt{n}})).$$

Noticing the fact that $\overline{V}_{n,2}(f,x)$ have the same degree of approximation as $V_{n,2}(f,x)$, and the degree of trigonometric polynomial $\overline{V}_{n,2}(f,x)$ is $3n-1$, we say that $\overline{V}_{n,2}(f,x)$ are better than $V_{n,2}(f,x)$ for real application.

We also consider the approximation of combination of Cesáro means $\sigma_n^a(f,x)$ with order $a (a > 0)$, Fejér -Korovkin operators $K_n(f,x)$ and Jackson operators $J_n(f,x)$. Write

$$\sigma^{a;q}_{n;m}(f,x) = A_m^{(m)} \sigma^a_{q_m n}(f,x) + \cdots + A_1^{(m)} \sigma^a_{q_1 n}(f,x) + A_0^{(m)} \sigma^a_n(f,x),$$

$$K^q_{n,m}(f,x) = B_m^{(m)} K_{q_m n}(f,x) + \cdots + B_1^{(m)} K_{q_i n}(f,x) + B_0^{(m)} K_n(f,x),$$

$$J^q_{n,m}(f,x) = B_m^{(m)} J_{q_m n}(f,x) + \cdots + B_1^{(m)} J_{q_i n}(f,x) + B_0^{(m)} J_n(f,x),$$

where numbers $B_0^{(m)}, B_1^{(m)}, \cdots B_m^{(m)}$ satisfy the following equation system:

$$B_0^{(m)} + B_1^{(m)} + \cdots + B_m^{(m)} = 1,$$

$$B_0^{(m)} + \frac{1}{q_1^2} B_1^{(m)} + \cdots + \frac{1}{q_m^2} B_m^{(m)} = 0,$$

… …

$$B_0^{(m)}+\frac{1}{q_1^{m+1}}B_1^{(m)}+\cdots+\frac{1}{q_m^{m+1}}B_m^{(m)}=0.$$

We have

Theorem 3 Let $a>0$, $f(x)\in C_{2\pi}$ and $\tilde{f}(x)\in C_{2\pi}$. Then

$$\sigma^{a,q}_{n,m}(f,x)-f(x)=O\left(\omega_{m+1}\left(f,\frac{1}{n}\right)+\omega_{m+1}\left(\tilde{f},\frac{1}{n}\right)\right)$$

Theorem 4 Let $f(x)\in C_{2\pi}$, $\tilde{f}(x)\in C_{2\pi}$. Then

$$K^q_{n,m}(f,x)-f(x)=O\left(\omega_{m+2}\left(f,\frac{1}{n}\right)+\omega_{m+2}\left(\tilde{f},\frac{1}{n}\right)\right).$$

Theorem 5 Let $f(x)\in C_{2\pi}$. Then

$$J^q_{n,m}(f,x)-f(x)=O\left(\omega_{m+2}\left(f,\frac{1}{n}\right)+\omega_{m+2}\left(\tilde{f},\frac{1}{n}\right)\right).$$

Specially, writing

$$\sigma^a_{n,1}(f,x)=2\sigma^a_{2n}(f,x)-\sigma^a_n(f,x),$$

$$\sigma^a_{n,2}(f,x)=\frac{9}{2}\sigma^a_{3n}(f,x)-4\sigma^a_{2n}(f,x)+\frac{1}{2}\sigma^a_n(f,x),$$

$$K_{n,1}(f,x)=\frac{4}{3}K_{2n}(f,x)-\frac{1}{3}K_n(f,x),$$

$$K_{n,2}(f,x)=\frac{9}{4}K_{3n}(f,x)-\frac{4}{3}K_{2n}(f,x)+\frac{1}{12}K_n(f,x),$$

$$J_{n,1}(f,x)=\frac{4}{3}J_{2n}(f,x)-\frac{1}{3}J_n(f,x),$$

$$J_{n,2}(f,x)=\frac{9}{4}J_{3n}(f,x)-\frac{4}{3}J_{2n}(f,x)+\frac{1}{12}J_n(f,x),$$

we have

Corollary 2 Let $a>0$, $f(x)\in C_{2\pi}$, $\tilde{f}(x)\in C_{2\pi}$. Then $\sigma^a_{n,1}(f,x)-f(x)=O(\omega_2(f,\frac{1}{n})+\omega_2(\tilde{f},\frac{1}{n}))$, $\sigma^a_{n,2}(f,x)-f(x)=O(\omega_3(f,\frac{1}{n})+\omega_3(\tilde{f},\frac{1}{n}))$.

Corllary 3 Let $f(x)\in C_{2\pi}$. Then $K_{n,1}(f,x)-f(x)=O(\omega_3(f,\frac{1}{n})+\omega_3(\tilde{f},\frac{1}{n}))$, $K_{n,2}(f,x)-f(x)=O(\omega_4(f,\frac{1}{n})+\omega_4(\tilde{f},\frac{1}{n}))$.

Corollory 4 Let $f(x)\in C_{2\pi}$. Then $J_{n,1}(f,x)-f(x)=O(\omega_3(f,\frac{1}{n})+\omega_3(\tilde{f},\frac{1}{n}))$, $J_{n,2}(f,x)-f(x)=O(\omega_4(f,\frac{1}{n})+\omega_4(\tilde{f},\frac{1}{n}))$.

Before we give the proof of theorem 2, we frist establish the following

Theorem I Suppose that $f(x)\in C_{2\pi}$, $f^{(2m+2)}(x)\in C_{2\pi}$. Then $V^q_{n,m}(f,x)-f(x)=$

$O(M(f)n^{-m-1})$, where $M(f)=\max\limits_{1\le i\le m+1}\|f^{(2i)}(x)\|$.

Proof From[2], we know that in this case holds the following asymptotic expansion

$$V_{n-1}(f,x)-f(x)=\sum_{j=1}^{m}\left[\frac{(-1)^j}{(2j)!}\sum_{q=2j}^{2m}C_{2j,q-2j}\Delta^q\rho_0^{(n-1)}\right]f^{(2j)}(x) \tag{2}$$
$$+O(\|f^{(2m+2)}(x)\|n^{-m-1}),$$

where $\Delta^q\rho_0^{(n-1)}=\sum\limits_{k=0}^{q}(-1)^{q-k}\binom{q}{k}\rho_k^{(n-1)}$, $\rho_k^{(n-1)}=\dfrac{[(n-1)!]^2}{(n-1+k)!(n-1-k)!}$

Since

$$\rho_k^{(n-1)}=\frac{(n-1)(n-2)\cdots(n-k)}{(n-1+k)(n-2+k)\cdots(n+1)n}=\frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k}{n})}{(1+\frac{k-1}{n})(1+\frac{k-2}{n})\cdots(1+\frac{1}{n})}$$

and, for $1\le k\le 2m$,

$$\frac{1}{1+\frac{k-1}{n}}=1-\frac{k-1}{n}+\left(\frac{k-1}{n}\right)^2-\cdots+(-1)^m\left(\frac{k-1}{n}\right)^m+O(n^{-m-1}),$$
$$\frac{1}{1+\frac{k-2}{n}}=1-\frac{k-2}{n}+\left(\frac{k-2}{n}\right)^2-\cdots+(-1)^m\left(\frac{k-2}{n}\right)^m+O(n^{-m-1}),$$
$$\cdots\qquad\cdots$$
$$\frac{1}{1+\frac{1}{n}}=1-\frac{1}{n}+\left(\frac{1}{n}\right)^2-\cdots+(-1)^m\left(\frac{1}{n}\right)^m+O(n^{-m-1}),$$

we can have

$$\rho_k^{(n-1)}=1+D_{k,1}\frac{1}{n}+D_{k,2}\frac{1}{n^2}+\cdots+D_{k,m}\frac{1}{n^m}+O(n^{-m-1})\ (1\le k\le 2m),$$

where $D_{k,i}$ $(1\le i\le m)$ are independent of n. Thus formula (2) can be rewritten in the following form:

$$V_{n-1}(f,x)-f(x)=D_{m,1}(f,x)\frac{1}{n}+D_{m,2}(f,x)\frac{1}{n^2}+\cdots+D_{m,m}(f,x)\frac{1}{n^m}+$$
$$+O(M(f)n^{-m-1}), \tag{3}$$

where $D_{m,i}(f,x)$ $(1\le i\le m)$ are independent of n.

According to (1), $A_0^{(m)}+A_1^{(m)}+\cdots+A_m^{(m)}=1$. hence

$$V_{n,m}^q(f,x)-f(x)=A_m^{(m)}[V_{q_m n-1}(f,x)-f(x)]+\cdots+A_1^{(m)}[V_{q_1 n-1}(f,x)-f(x)]$$
$$+A_0^{(m)}[V_{n-1}(f,x)-f(x)].$$

Then, using formulae (3) and (1), we obtain

$$V_{n,m}^q(f,x)-f(x)=A_m^{(m)}[D_{m,1}(f,x)\frac{1}{q_m\cdot n}+D_{m,2}(f,x)\frac{1}{q_m^2n^2}+\cdots+D_{m,m}(f,x)\cdot$$
$$\frac{1}{q_m^mn^m}]+\cdots+A_1^{(m)}[D_{m,1}(f,x)\frac{1}{q_1n}+D_{m,2}(f,x)\frac{1}{q_1^2n^2}+\cdots+D_{m,m}(f,x)\frac{1}{q_1^mn^m}]$$

$$+A_0^{(m)}[D_{m,1}(f,x)\frac{1}{n}+D_{m,2}(f,x)\frac{1}{n^2}+\cdots+D_{m,m}(f,x)\frac{1}{n^m}]+O(M(f)n^{-m-1})$$

$$=D_{m,1}(f,x)\frac{1}{n}\cdot[A_m^{(m)}\frac{1}{q^m}+\cdots+A_1^{(m)}\frac{1}{q}+A_0^{(m)}]+D_{m,2}(f,x)\frac{1}{n^2}\cdot[A_m^{(m)}\frac{1}{q_m^2}+\cdots+$$

$$A_1^{(m)}\frac{1}{q_1^2}+A_0^{(m)}]+\cdots+D_{m,m}(f,x)\frac{1}{n^m}\cdot[A_m^{(m)}\cdot\frac{1}{q_m^m}+\cdots+A_1^{(m)}\frac{1}{q_1^m}+A_0^{(m)}]$$

$$+O(M(f)n^{-m-1})=O(M(f)n^{-m-1}).$$

This completes the proof of theorem 1.

Proof of Theorem 2 It is explicit that the norms of the operators $V_{n,m}^q(f,x)$ are bounded uniformly with respect to n. Suppose that $t_n(f,x)$ are the trigonometric polynomials of best approximation of $f(x)$ wtih degree $[\sqrt{n}]$. Then, we have

$$V_{n,m}^q(f,x)-f(x)=V_{n,m}^q(t_n,x)-t_n(f,x)+O(E_{[\sqrt{n}]}(f)).$$

Applying theorem 1 to $t_n(f,x)$, we obtain

$$V_{n,m}^q(t_n,x)-t_n(f,x)=O(M(t_n)n^{-m-1})=O(\max_{1\le i\le m+1} n^{-m-1+i}\omega_{2i}(t_n,\frac{1}{\sqrt{n}}))$$

$$=O(\omega_{2m+2}(f,\frac{1}{\sqrt{n}})).$$

Thus holds $V_{n,m}^q(f,x)-f(x)=O(\omega_{2m+2}(f,\frac{1}{\sqrt{n}}))$. The proof is complete.

Using the asymptotic expansions of $\sigma_n^q(f,x)$, $K_n(f,x)$ and $J_n(f,x)$ in [3], [4] and [5], we can prove theorem 3, 4 and 5 similarly. The proofs are omitted here.

References

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