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An Extension of the Turan Inequality in L_p-space for $0<p<1$ *

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1. Introduction

Denote by H_n the set of n-th algebraic polynomials all whose roots lie inside $[-1, 1]$, R_n is the class of algebraic polynomials of degree n which have only real roots, and R_n^* is the class of trigonometric polynomials of degree n having only real zeros. Let C be a positive absolute constant, which may be different in different places.

Theorem T. If $f(x)\in H_n$, then

$$\|f'\|_{L_p[-1,1]}\geqslant C\sqrt{n}\|f\|_{L_p[-1,1]} \qquad (1<p<\infty), \tag{1}$$

where $\|\cdot\|_{L_p[-1,1]}$ is the L_p-norm on $[-1, 1]$.

In 1939, P. Turán[1] proved the inequality (1) in L_∞-norm, later, A. K. Varma[2] Extended it to L_2-space, and recently, in[3], we obtained the similar result in L_p-space $(1\leqslant p<\infty)$.

Now we ask whether it is valid that $\int_{-1}^1|f'(x)|^p dx\geqslant C\sqrt{n}^p\int_{-1}^1|f(x)|^p dx$ for $f(x)\in H_n$ and $0<p<1$? This paper will give a positive answer to this question.

2. Some Lemmas

We first establish some lemmas. Let $-1\leqslant x_1<x_2<\cdots<x_s\leqslant 1$ be all roots of $f(x)\in H_n$, x_i a root of order l_i, a_i a maximun point of $|f(x)|$ in (x_i, x_{i+1}), and $p\in(0,1)$. In the following lemmas, we consider only the case of the interval (x_i, a_i), but we must point out that there are similar results corresponding to the interval (a_i, x_{i+1}).

Lemma 1. Let $l(x)=\sum_{k=1}^s\frac{l_k}{x-x_k}$.

(i) If $a_i-x_i\leqslant\frac{1}{\sqrt{n}}$, then there exists ξ_i (x_i, a_i) satisfying $\xi_i\geqslant\frac{a_i+2x_i}{3}$ such that

$$|l(x)|\geqslant\frac{2}{3}\sqrt{n}, \text{ for } x\ (x_i, \xi_i], \tag{2}$$

$$|l(x)|\leqslant\frac{2}{3}\sqrt{n}, \text{ for } x\ [\xi_i, a_i]. \tag{3}$$

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(ii) If $a_i - x_i > \frac{1}{\sqrt{n}}$, then there exists $\xi_i' \in (x_i, a_i)$ satisfying $\xi_i' \geqslant a_i - \frac{1}{2\sqrt{n}}$ such that

$$|l(x)| > \frac{\sqrt{n}}{8} \text{ for } x \in (x_i, \xi_i'], \tag{4}$$

$$|l(x)| \frac{\sqrt{n}}{8} \text{ for } x \in [\xi_i', a_i], \tag{5}$$

Proof. (i) Obviously, $l(x) = \frac{f'(x)}{f(x)}$, and $l(x)$ is continuous and decreasing in $(x_i, a_i]$, $\lim_{x \to x_{i,0}} l(x) = +\infty$, $l(a_i) = 0$, then there exists $\xi_i \in (x_i, a_i)$ such that $l(\xi_i) = \frac{2}{3}\sqrt{n}$, hence the monotonicity of $l(x)$ implies (2) and (3). We prove that $\xi_i \geqslant \frac{a_i + 2x_i}{3}$ as follows.

It is easily seen that $l(x) = \int_x^{a_i} \sum_{k=1}^{s} \frac{l_k}{(t - x_k)^2} dt$ for $x \in (x_i, a_i]$, therefore

$$l\left(\frac{a_i + 2x_i}{3}\right) = \int_{\frac{a_i+2x_i}{3}}^{a_i} \sum_{k=1}^{s} \frac{l_k}{(t - x_k)^2} dt \geqslant \int_{\frac{a_i+2x_i}{3}}^{\frac{a_i+x_i}{2}} \frac{1}{(t - x_i)^2} dt \geqslant \frac{4}{(a_i - x_i)^2} \frac{a_i - x_i}{6}$$

$= \frac{2}{3} \frac{1}{a_i - x_i}$, from $a_i - x_i \leqslant \frac{1}{\sqrt{n}}$ we get $l\left(\frac{a_i + 2x_i}{3}\right) \geqslant \frac{2}{3}\sqrt{n}$, that is $\xi_i \geqslant \frac{a_i + 2x_i}{3}$.

(ii) Similarly, there exists such $\xi_i' \in (x_i, a_i)$ that (4) and (5) are valid, besides $\left|l\left(a_i - \frac{1}{2\sqrt{n}}\right)\right| = \left|l\left(a_i - \frac{1}{2\sqrt{n}}\right) - l(a_i)\right| = \sum_{k=1}^{s} \frac{l_k}{(\delta - x_k)^2} \frac{1}{2\sqrt{n}}$, where $\delta \in \left[a_i - \frac{1}{2\sqrt{n}}, a_i\right]$.

From $\sum_{k=1}^{s} \frac{l_k}{(\delta - x_k)^2} \geqslant \frac{n}{4}$ we get $l\left(a_i - \frac{1}{2\sqrt{n}}\right) \geqslant \frac{\sqrt{n}}{8}$, and it follow that $\xi_i' \geqslant a_i - \frac{1}{2\sqrt{n}}$ from the monotonicity of $l(x)$ in $(x_i, a_i]$. ■

Lemma 2. Suppose that ξ_i, ξ_i' are numbers in Lemma 1, then the following inequalities are true:

$$l\left(\frac{2a_i + \xi_i}{3}\right) \geqslant \frac{1}{19} l(\xi_i), \tag{6}$$

$$l\left(\frac{2a_i + \xi_i'}{3}\right) \geqslant \frac{1}{9} l(\xi_i'). \tag{7}$$

Proof. We first prove (6). Assume that $l(\xi_i) - l\left(\frac{2a_i + \xi_i}{3}\right) = \frac{2}{3}(a_i - \xi_i) \cdot \sum_{k=1}^{s} \frac{l_k}{(\sigma - x_k)^2}$, where $\sigma \in \left[\xi_i, \frac{2a_i + \xi_i}{3}\right]$; and $l\left(\frac{2a_i + \xi_i}{3}\right) = \frac{a_i - \xi_i}{3} \cdot \sum_{k=1}^{s} \frac{l_k}{(\eta - x_k)^2}$ where $\eta \in \left[\frac{2a_i + \xi_i}{3}, a_i\right]$. For $k < i$, it follows from Lemma 1 that $\sigma - x_k \geqslant \xi_i - x_i \geqslant \frac{1}{3}(a_i - x_i)$. Then $\eta - x_k = \eta - \sigma + \sigma - x_k \leqslant a_i - \xi_i + \sigma - x_k \leqslant 3(\sigma - x_k)$. For $k > i$, it is clear that $|\eta - x_k| \leqslant |\sigma - x_k|$. Hence $l\left(\frac{2a_i + \xi_i}{3}\right) = \frac{a_i - \xi_i}{3} \sum_{k=1}^{s} \frac{l_k}{(\sigma - x_k)^2} \frac{(\sigma - x_k)^2}{(\eta - x_k)^2} \geqslant \frac{a_i - \xi_i}{3} \cdot \sum_{k=1}^{s} \frac{l_k}{(\sigma - x_k)^2} \frac{1}{9} = \frac{1}{18}\left(l(\xi_i) - l\left(\frac{2a_i + \xi_i}{3}\right)\right)$, that is $l\left(\frac{2a_i + \xi_i}{3}\right) \geqslant \frac{1}{19} l(\xi_i)$.

The proof of the inequality (7) is similar to that of (6). ■

Lemma 3. If $a_i - x_i \leqslant \frac{1}{\sqrt{n}}$, then $\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{x_i}^{a_i} |f(x)|^p dx$.

Proof. We consider the following two cases.

Case 1. $|f(\xi_i)| \geqslant \frac{1}{2}|f(a_i)|$. Taking into account that $\int_{x_i}^{a_i} |f'(x)|^p dx$ $= \int_{x_i}^{a_i} |f(x)|^p |l(x)|^p dx$, and from Lemma 1 and Lemma 2 we get

$$\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant \left(\frac{2\sqrt{n}}{57}\right)^p \int_{x_i}^{\frac{2a_i-\xi_i}{3}} |f(x)|^p dx. \qquad (8)$$

Since $|f(\xi_i)| \geqslant \frac{1}{2}|f(a_i)|$, $\int_{\xi_i}^{\frac{2a_i+\xi_i}{3}} |f(x)|^p dx \geqslant \left(\frac{1}{2}\right)^p \int_{\frac{2a_i+\xi_i}{3}}^{a_i} |f(x)|^p dx$.

Combine it with (8), we have

$$\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant \frac{1}{2}\left(\frac{1}{57}\right)^p \sqrt{n}^p \int_{x_i}^{a_i} |f(x)|^p dx.$$

Case 2. $|f(\xi_i)| < \frac{1}{2}|f(a_i)|$. From (3) we get

$$|f'(x)| = |f(x)|\,|l(x)| \leqslant |f(a_i)|\frac{2}{3}\sqrt{n}$$

for $x \in [\xi_i, a_i]$. Hence

$$\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant \int_{\xi_i}^{a_i} |f'(x)|^{p-1}|f'(x)| dx \geqslant \left(\frac{2}{3}\sqrt{n}\right)^{p-1} |f(a_i)|^{p-1} \int_{\xi_i}^{a_i} |f'(x)| dx$$

$$= \left(\frac{2}{3}\sqrt{n}\right)^{p-1} |f(a_i)|^{p-1} |f(a_i) - f(\xi_i)|.$$

Combining it with $|f(\xi_i)| < \frac{1}{2}|f(a_i)|$ and $a_i - x_i \leqslant \frac{1}{\sqrt{n}}$, we have

$$\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant \frac{1}{2}\left(\frac{2}{3}\right)^{p-1} \sqrt{n}^{p-1} |f(a_i)|^p \geqslant \frac{1}{2}\left(\frac{2}{3}\right)^{p-1} \sqrt{n}^p \int_{x_i}^{a_i} |f(x)|^p dx. \ ■$$

Lemma 4. If $a_i - x_i > \frac{1}{\sqrt{n}}$, then $\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{x_i}^{a_i} |f(x)|^p dx$.

Proof. Obviously $\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant \left(\frac{\sqrt{n}}{72}\right)^p \int_{x_i}^{\frac{2a_i+\xi_i'}{3}} |f(x)|^p dx$, similar to the proof of Lemma 3, we have $\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{\xi_i'}^{a_i} |f(x)|^p dx$, thus we complete the proof of Lemma 4. ■

3. Main Results.

Theorem 1. If $f(x) \in H_n$, then

$$\int_{-1}^{1} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{-1}^{1} |f(x)|^p dx \qquad (0 < p < 1).$$

Proof. From $\int_{-1}^{x_1} |f'(x)|^p dx = \int_{-1}^{x_1} |f(x)|^p |l(x)|^p dx$, we get easily that

$$\int_{-1}^{x_1} |f'(x)|^p dx \geqslant \left(\frac{n}{2}\right)^p \int_{-1}^{x_1} |f(x)|^p dx. \tag{9}$$

Similarly,

$$\int_{x_s}^{1} |f'(x)|^p dx \geqslant \left(\frac{n}{2}\right)^p \int_{x_s}^{1} |f(x)|^p dx. \tag{10}$$

Taking into account Lemma 3 and Lemma 4 , it follows that

$$\int_{x_i}^{a_i} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{x_i}^{a_i} |f(x)|^p dx. \tag{11}$$

Similarly,

$$\int_{a_i}^{x_{i+1}} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{a_i}^{x_{i+1}} |f(x)|^p dx . \tag{12}$$

We write

$$\int_{-1}^{1} |f'(x)|^p dx = \int_{-1}^{x_1} |f'(x)|^p dx + \int_{x_s}^{1} |f'(x)|^p dx + \sum_{k=1}^{s-1} \int_{x_k}^{a_k} |f'(x)|^p dx$$
$$+ \sum_{k=1}^{s-1} \int_{a_k}^{x_{k+1}} |f'(x)|^p dx ,$$

and from (9)—(12) we get $\int_{-1}^{-1} |f'(x)|^p dx \geqslant C\sqrt{n}^p \int_{-1}^{1} |f(x)|^p dx.$

Taking $f(x) = (1 - x^2)^k$, $k = [\frac{n}{2}]$, we have

$$\int_{-1}^{1} |f(x)|^p dx \sim Cn^{\frac{1}{2}}, \quad \int_{-1}^{1} |f'(x)|^p dx \sim Cn^{\frac{p-1}{2}},$$

and from this we easily see that the order of n in Theorem 1 cannot be improved. Applying similar means. the result above can be silghtly extended to

Theorem 2. If $f(x) \in R_n$ has at most k roots lying outside $[-1, 1]$, then $\int_{-1}^{1} |f'(x)|^p dx \geqslant C_k\sqrt{n^p} \int_{-1}^{1} |f(x)|^p dx$. where C_k is a positive constant only depending on k, $0 < p < 1$.

For periodic case, we have the similar result:

Theorem 3 . If $t(x) \in R_n^*$, then

$$\int_{-\pi}^{\pi} |t'(x)|^p dx \geqslant C\sqrt{n}^p \int_{-\pi}^{\pi} |t(x)|^p dx \qquad (0 < p < 1) .$$

Taking $t(x) = (\cos\frac{x}{2})^{2n}$, it is easily seen that the order of n of the above expression cannot be improved.

References

[1] Turan P. , Über die Ableitung von Polynomen, Compositio Math., 7 (1939), 89—95.

[2] Varma, A. K., An analogue of some inequalities of P. Turán concerning algebraic polynomials satisfying certain conditions, Proc. Amer. Math. Soc., 55 (1976), 305—309.

[3] 周颂平，关于L_p尺度下的Turan 不等式，杭州大学学报（自然科学版），待发表。