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A Note On The Kalman Canonical Decomposition*

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Decomposing a linear dynamic system

$$\begin{cases} \dot{x} = Ax + Bu \\ v = Cx \end{cases} \tag{1}$$

where $x \in R^n$, $u \in R^p$, and $v \in R^q$ with $p \leqslant n$ and $q \leqslant n$, means reformulating it as a joint controllability-observability structure by using a nonsingular transformation $y = Qx$, namely:

$$\begin{cases} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u, \\ \\ v = [\,0 \;\; C_2 \;\; 0 \;\; C_4\,] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \end{cases} \tag{2}$$

where $y = [y_1 y_2 y_3 y_4]^T$ with y_1 being an $n_1 \times 1$ state vector in the first subsystem which is completely controllable but not observable, y_2 being an $n_2 \times 1$ state vector in the second subsystem which is both completely controllable and observable, y_3 being an $n_3 \times 1$ state vector in the third subsystem which is neither controllable nor observable, y_4 being an $n_4 \times 1$ state vector in the fourth subsystem which is observable but not controllable, and $n_1 + n_2 + n_3 + n_4 = n$.

This problem was first considered in Gilbert [4] under the assumption that the eigenvalues of the system matrix A are distinct. A generalization to time-varying systems was studied in Kalman [5, 6] and later in Weiss [8]. This joint controllability-observability structure is now called a Kalman canonical decomposition.

The purpose of this note is to point out that so far no complete proof to this decom-

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position structure has been pubilshed even for time-invariant systems.

Let M denote the (n_1+n_2)-dimensional linear subspace consisting of all controllable state vectors and N the (n_1+n_3)-dimensional linear subspace consisting of all unobservable state vectors in the time-invariant system (1). Then both M and N are invariant subspaces unber A and consequently the n_1-dimensional subspace $M \cap N$ is also invariant under A. Kalman claimed (cf. [6]) that the decomposition structure (2) follows immediately from this invariance properties. More precisely, what he meant is the following: since

$$\begin{bmatrix} y_1 \\ y_2 \\ * \\ * \end{bmatrix} \in M, \quad \begin{bmatrix} y_1 \\ * \\ y_3 \\ * \end{bmatrix} \in N, \text{ and } \begin{bmatrix} y_1 \\ * \\ * \\ * \end{bmatrix} \in M \cap N,$$

the state vector y_1 in the first subsystem must be controllable and unobservable. It will be seen from the following counterexample that this is not true! The invariance of $M \cap N$ under A alone does not guarantee the controllability of the state vector y_1. In fact, Boley [1] has given a counterexample to show that Kalman's decomposition precedure doesn't work, but did not point out the neglegence in Kalman's proof.

Other approaches have been investigated. For instance, Fortmann and Hitz [3] used a unitary transformation to decompose the state space to a direct sum of four subspaces and Sun [7] used another nonsingular transformation to decompose the state space. The fact that the former approach cannot give the desired decomposition structure is known to many experts in this area (cf., for example, Sun [7]). A simple example is the system

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } C = [0 \ 1 \ 0 \ 1].$$

As it stands, this is already in the desired decomposed form with $n_1 = n_2 = n_3 = n_4 = 1$. The second subsystem is clearly both completely controllable and observable, and the combined subsystem $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $C = [0 \ 1]$ is also completely controllable. However, the first subsystem is not controllable. Moreover, it can be easily shown that any unitary transformation cannot change the first subsystem to be controllable (cf. Chui and Chen [2] for more detail).

In [7], Sun claimed that a proof to the decomposition structure has been obtained. Since his proof was also based on the invariance properties, it did not give more than what Kalman did, and unfortunately, is also wrong. To show this, let us first sketch his proof (a constructive procedure) as folows.

Using the notation in [7], let

$$\mathrm{R}_c = \mathrm{sp}\{ B\ AB \cdots A^{n-1}B\}$$

be the controllable subspace and N_c the uncontrollable subspace of the state space associated with the time-invariant system (1) such that $\mathrm{R}_c \oplus \mathrm{N}_c = \mathrm{R}^n$. Similarly, let

$$\mathrm{N}_0 = \mathrm{sp}\{ x \in \mathrm{R}^n : \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0 \}$$

be the unobservable subspace and R_0 the observable subspace with $\mathrm{R}_0 \oplus \mathrm{N}_0 = \mathrm{R}^n$. Then R_c and N_0, and consequently $\mathrm{R}_c \cap \mathrm{N}_0$, are invariant subspaces under A. Note that according to Kalman or Sun, any state vector x in $\mathrm{R}_c \cap \mathrm{N}_0$ should be controllable and unobservable. Sun's procedure is as forllows:

Step 1: Choose N_c such that $\mathrm{N}_c \oplus \mathrm{R}_c = \mathrm{R}^n$ and $\mathrm{N}_c \cap \mathrm{N}_0 \neq \phi$. Denote $\dim(\mathrm{N}_c \cap \mathrm{N}_0) = n_3$.

Step 2: Choose R_0 such that $\mathrm{R}_0 \oplus \mathrm{N}_0 = \mathrm{R}^n$, $\mathrm{R}_0 \cap \mathrm{R}_c \neq \phi$ and $\mathrm{R}_0 \cap \mathrm{N}_c \neq \phi$. Denote $\dim(\mathrm{R}_0 \cap \mathrm{R}_c) = n_2$ and $\dim(\mathrm{R}_0 \cap \mathrm{N}_c) = n_4$.

Step 3: Denote $\mathrm{R}_1 = \mathrm{R}_c \cap \mathrm{N}_0 = \mathrm{sp}\{\mathrm{r}_1, \cdots, \mathrm{r}_{n_1}\}$, $\dim(\mathrm{R}_1) = n_1$,

$\mathrm{R}_2 = \mathrm{R}_c \cap \mathrm{R}_0 = \mathrm{sp}\{\mathrm{r}_{n_1+1}, \cdots, \mathrm{r}_{n_1+n_2}\}$, $\dim(\mathrm{R}_2) = n_2$,

$\mathrm{R}_3 = \mathrm{N}_c \cap \mathrm{N}_0 = \mathrm{sp}\{\mathrm{r}_{n_1+n_2+1}, \cdots, \mathrm{r}_{n_1+n_2+n_3}\}$, $\dim(\mathrm{R}_3) = n_3$,

$\mathrm{R}_4 = \mathrm{N}_c \cap \mathrm{R}_0 = \mathrm{sp}\{\mathrm{r}_{n_1+n_2+n_3+1}, \cdots, \mathrm{r}_{n_1+n_2+n_3+n_4}\}$, $\dim(\mathrm{R}_4) = n_4$.

Then $n_1 + n_2 + n_3 + n_4 = n$ and $\mathrm{R}_1 \oplus \mathrm{R}_2 \oplus \mathrm{R}_3 \oplus \mathrm{R}_4 = \mathrm{R}^n$. Let the $n \times n$ nonsingular transformation matrix Q be

$$Q = [\mathrm{r}_1 \cdots \mathrm{r}_{n_1+n_2+n_3+n_4}] .$$

Then, based on the invariance properties mentioned above, the transformation $y = Qx$ gives the decomposition structure (2) of system (1).

The following counterexample shows that this procedure does not work either. Consider the system with

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } C = [0\ 1\ 1] .$$

We have

$$[B\ AB\ \ A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

Hence, this system is completely controllable but not observable. Also

$$N_0 = \mathrm{sp}\{ x \in R^3 : \begin{bmatrix} C \\ CC \\ CC \end{bmatrix} x = 0 \} = \mathrm{sp}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$$

and

$$R_c = \mathrm{sp}\{ B\ AB\ A^2B \} = \mathrm{sp}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \} = R^3 .$$

Step 1: Since $R_c = R^3$, We must have $N_c = \{0\}$ so that $n_3 = \dim(N_c \cap N_0) = 0$, $n_4 = \dim(R_0 \cap N_c) = 0$, and $n_1 = \dim(R_c \cap N_0) = 1$.

Step 2: Choose R_0 such that $R_0 \oplus N_0 = R^3$ and $\dim(R_0 \cap R_c) = n_2 = n - n_1 = 2$. We may choose

$$R_0 = \mathrm{sp}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}.$$

Then $R_0 \oplus N_0 = \mathrm{sp}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} = R^3$ and $\dim(R_0 \cap R_c) = \dim(\mathrm{sp}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}) = 2$

Step 3: It follows that

$$R_1 = R_c \cap N_0 = \mathrm{sp}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \},\ n_1 = 1, \quad R_2 = R_c \cap R_0 = \mathrm{sp}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \},\ n_2 = 2,$$

$$R_3 = N_c \cap N_0 = \{0\},\ n_3 = 0, \quad R_4 = N_c \cap R_0 = \{0\},\ n_4 = 0.$$

Let $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Then we have $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = Q^{-1} A Q = \left[\begin{array}{c:cc} 1 & 0 & 1 \\ \hdashline 0 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right],$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = Q^{-1} B = \left[\begin{array}{c} 0 \\ \hdashline 1 \\ 0 \end{array}\right], \text{ and } [C_1 C_2] = CQ = [0 \,\vdots\, 1\ 2].$$

We have carefully followed Sun's procedure here and it turns out that the first subsystem is neither controllable nor observable although R_1 is the intersection of the controllable subspace R_c and the unobservable subspace N_0.

Finally, it should be remarked that our counterexample does not disprove the decomposition structure (2). Indeed, since we have infinitely many choices of R_0, it is quite possible that a suitable one can do the job. In this example, for instance, choosing

$$R_0 = \mathrm{sp}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\},$$

we have

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

so that the desired system is obtained, namely:

$$\begin{cases} \dot{y} = \left[\begin{array}{c:cc} 1 & 0 & 1 \\ \hdashline 0 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] y + \begin{bmatrix} 1 \\ \cdots \\ 2 \\ -1 \end{bmatrix} u \\ v = [\,0 \;\vdots\; 2 \;\; 3\,]\, y \end{cases}$$

In Sun [7], however, the choice of a suitable R_0 was not mentioned. This is in fact the key step to achieving a correct proof to the Kalman canonical decomposition theorem. To the best of our knowledge, this approach does not seem to be easier than proving the result directly.

In summary, a rigorous proof of the Kalman canonical decomposition theorem is still unavailable in the literature.

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关于Kalman的标准系统分解

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本文给出一个反例，指出Kalman关于他本人提出的标准系统分解的原始证明以及后来孙承启先生在《自动化学报》1984年第3期上发表的证明（参见〔7〕）均是错误的。事实上，如何严格地证明（或反证）这一重要结论尚是一个未有彻底解决的问题。