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On Trigonometric Splines*

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A great many papers concerning polynomial spline have been published in various mathematical journals. But, as we know, there are a few papers discussing trigonometric splines. The aim of this paper is to investigate some kinds of trigonometric splines, which can be regarded as a generalization of the polynomial splines discussed in [1], [2]. From the results obtained in this paper we will see the similarity and differences between them.

Suppose $\Delta_n: 0=x_0<x_1<\cdots<x_n=1$ is a partition of the interval $[0,1]$, $h_\nu=x_{\nu+1}-x_\nu$, $\nu=0,1,\cdots,n-1$, $t_{\nu,i}=x_\nu+\frac{i}{2m}h_\nu$, $\nu=0,1,\cdots,n-1$, $i=0,1,\cdots,2m$, $f'(x)\in C[0,1]$, If $T^*_{\Delta_n}(f;x)=T^*_{\Delta_n}(x)$ satisfies

(i) In each $[x_\nu,x_{\nu+1}]$, $\nu=0,1,\cdots,n-1$, $T^*_{\Delta_n}(f;x)$ is a trigonometric polynomial of degree $\leqslant m+1$.

(ii) $T^*_{\Delta_n}(f;x)\in C^2[0,1]$.

(iii) $T^*_{\Delta_n}(f;t_{\nu,i})=f(t_{\nu,i})$, $\nu=0,1,\cdots,n-1$, $i=0,1,\cdots,2m$,

then, we say $T^*_{\Delta_n}(x)$ is an interpolating trigonometric spline of $f(x)$ with x_ν as spline knots and $t_{\nu,i}$ interpolating nodes.

To determine $T^*_{\Delta_n}(x)$ uniquely, we need end conditions

(iv) $T^*_{\Delta_n}(f;0)=f'(0)$, $T^{*\prime}_{\Delta_n}(f;1)=f'(1)$.

Those splines with end conditions can be discussed by same method and we wouldn't repeat.

First of all, we discuss Hermite interpolating trigonometric spline $T^*_{\Delta_n}(f;x)$, which satisfies

(i) In each $[x_\nu,x_{\nu+1}]$, $\nu=0,1,\cdots,n-1$, $T_\Delta(f;x)$ is a trigonometric polynomial of degree $\leqslant m+1$,

(ii) $T_{\Delta_n}(f;t_{\nu,i})=f(t_{\nu,i})$, $i=0,1,\cdots,2m$, $i=0,1,\cdots,n-1$, $T'_{\Delta_n}(f;x_\nu)=f'(x_\nu)$, $\nu=0,1,\cdots,n$.

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Suppose $u_{\nu,i,j}(x), \nu=0,1,\cdots, n-1, i=0,1,\cdots, 2m, j=0,1$ are functions satisfying the following conditions:

(i) when $x \in [x_\nu, x_{\nu+1}]$, $u_{\nu,i,j}(x) \equiv 0$,

(ii) in each $[x_\nu, x_{\nu+1}]$, $u_{\nu,i,j}(x)$ is a trigonometric polynomial of degree $\leqslant m+1$,

(iii) let $r_0 = r_{2m} = 1$, $r_1 = r_2 = \cdots = r_{2m-1} = 0$, $u_{\nu,i,j}^{(l)}(t_{\nu,k}) = \delta_{i,k}\delta_{j,l}$, where $i, k = 0, 1, \cdots, 2m$, $j, l \leqslant r_i$, $\nu = 0, 1, \cdots, n-1$,

(iv) $u_{\nu,i,j}(x) \equiv 0$, when $i = 1 > r_i$.

Then $T_{\Delta_n}(f; x)$ can be written as

$$T_{\Delta_n}(f; x) = \sum_{\nu=0}^{n-1}\sum_{i=0}^{2m}\sum_{j=0}^{1} f^{(j)}(t_{\nu,i})u_{\nu,i,j}(x) \tag{1}$$

We give some lemmas

Lemma 1 Let $T(x)$ be a trigonometric polynomial of degre $\leqslant m$, then

$$\max_{x\in[x_\nu, x_{\nu+1}]} |T'(x)| \leqslant c_m h_\nu^{-1} \max_{x\in[x_\nu, x_{\nu+1}]} |T(x)|,$$

where c_m is a constant depending only on m.

Proof Let $t_{\nu,i} = x_\nu + \frac{i}{2m}$, $i = 0, 1, \cdots, 2m$, $V_{\nu,i}(x) = \dfrac{\prod\limits_{\substack{j=0\\ j\neq i}}^{2m} 2\sin\frac{x - t_{\nu,j}}{2}}{\prod\limits_{\substack{j=0\\ j\neq i}}^{2m} 2\sin\frac{t_{\nu,i} - t_{\nu,j}}{2}}$

then

$$T(x) = \sum_{i=0}^{2m} T(t_{\nu,i})V_{\nu,j}(x).$$

Differentiating gives the lemma.

Lemma 2 When $x \in [x_\nu, x_{\nu+1}]$, $|u_{\nu,i,0}^{(g)}(x)| \leqslant C_m h_\nu^{-g}$, $|u_{\nu,i,1}^{(g)}(x)| \leqslant C_m h_\nu^{1-g}$

Proof From the definition, we have

$$u_{\nu,2m,1}(x) = a\prod_{i=1}^{2m}\left(2\sin\frac{x - t_{\nu,i}}{2}\right)\left(2\sin\frac{x - x_\nu}{2}\right)^2,$$

where $a = \left[\left(2\sin\frac{h_\nu}{2}\right)^2 \prod_{i=1}^{2m-1}\left(2\sin\frac{x_{\nu+1} - t_{\nu,i}}{2}\right)\right]^{-1}$, and the lemma can be obtained at once.

Combination of Lemmas 1 and 2 leads to

Lemma 3 When $x \in [x_\nu, x_{\nu+1}]$, $|u_{\nu,i,0}^{(q)}(x)| \leqslant c_m h_\nu^{-q}$, $|u_{\nu,i,1}^{(q)}(x)| \leqslant c_m h_\nu^{1-q}$.

Lemma 4 Suppose $f^{(r)}(x) \in C[0,1]$, $(1 \leqslant r \leqslant 2(m+1))$, then for $x \in [x_\nu, x_{\nu+1}]$, $q \leqslant r$,

$$|T_{\Delta_n}^{(q)}(x) - f^{(q)}(x)| \leqslant M h_\nu^{r-q}\omega(f^{(r)}; h_\nu) + M_2 h_\nu^{2m+3-q}\left(\sum_{p=0}^{r}\|f^{(p)}\|\right).$$

Proof Because of the developements $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, and $1-\cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$, it is easy to prove

$$x^p = \sum_{l=0}^{m+1} (a_{p,l}\cos lx + b_{p,l}\sin lx) + \sum_{l=2m+3}^{\infty} c_{p,l}x^l, \ (p = 1, 2, \cdots, \ 2(m+1)) \quad (2)$$

where $a_{p,j}$, $b_{p,j}$ are constants independent of x, and the power serieses converge on the whole interval $(-\infty, \infty)$

Thus, from the identity

$$p!\,\delta_{p,q} = [(t-x)^p]^{(q)}_{t=x} = [\sum_{l=0}^{m+1} \{a_{p,l}\cos l(t-x) + b_{p,l}\sin l(t-x)\}]^{(q)}_{t=x}$$
$$+ [\sum_{l=2m+3}^{\infty} c_{p,l}(t-x)^l]^{(q)}_{t=x},$$

it can be deduced

$$[\sum_{l=0}^{m+1} \{a_{p,l}\cos l(t-x) + b_{p,l}\sin l(t-x)\}]^{(q)}_{t=x} = p!\,\delta_{p,q} \ (0 \leqslant p, q \leqslant 2m+2). \quad (3)$$

By the definition of $u_{v,i,j}(x)$ and the fact that $T_{\Delta_n}(T; x) = T(x)$, if $T(x)$ is a trigonometric polynomial of degree $\leqslant m+1$, we have

$$p! \sum_{j=0}^{1} \sum_{i=0}^{2m} \frac{(t_{v,i}-x)^{p-j}}{(p-j)!} u_{v,i,j}(t) = \sum_{j=0}^{1} \sum_{i=0}^{2m} [(t-x)^p]^{(j)}_{t=t_{v,i}} u_{v,i,j}(t)$$
$$= \sum_{j=0}^{1} \sum_{i=0}^{2m} [\sum_{l=0}^{m+1} \{a_{p,l}\cos l(t-x) + b_{p,l}\sin l(t-x)\}]^{(j)}_{t=t_{v,i}} u_{v,i,j}(t)$$
$$+ \sum_{j=0}^{1} \sum_{i=0}^{2m} [\sum_{l=2m+3}^{\infty} c_{p,l}(t-x)^l]^{(j)}_{t=t_{v,i}} u_{v,i,j}(t)$$
$$= \sum_{l=0}^{m+1} a_{p,l} \sum_{j=0}^{1} \sum_{i=0}^{2m} [\cos l(t-x)]^{(j)}_{t=t_{v,i}} u_{v,i,j}(t)$$
$$+ \sum_{l=0}^{m+1} b_{p,l} \sum_{j=0}^{1} \sum_{i=0}^{2m} [\sin l(t-x)]^{(j)}_{t=t_{v,i}} u_{v,i,j}(t)$$
$$+ R_{p,m}(t,x) = \sum_{l=0}^{m+1} [a_{p,l}\cos l(t-x) + b_{p,l}\sin l(t-x)] + R_{p,m}(t,x), \quad (4)$$

where

$$R_{p,m}(t,x) = \sum_{j=0}^{1} \sum_{i=0}^{2m} [\sum_{l=2m+3}^{\infty} c_{p,l}(t-x)^l]^{(j)}_{t=t_{v,i}} u_{v,i,j}(t). \quad (5)$$

Substituting Taylor's developements

$$f^{(j)}(t_{v,i}) = \sum_{p=j}^{r} \frac{f^{(p)}(x)}{(p-j)!}(t_{v,i}-x)^{p-j} + \frac{f^{(r)}(\xi_{v,i,j}) - f^{(r)}(x)}{(r-j)!}(t_{v,i}-x)^{r-j}$$

into (1), we have, for $x \in [x_v, x_{v+1}]$,

$$T_{\Delta n}^{(q)}(x) \doteq \sum_{j=0}^{1} \sum_{i=0}^{2m} \left[\sum_{p=j}^{r} \frac{f^{(p)}(x)}{(p-j)!} (t_{v,i}-x)^{p-j} \right] u_{v,i,j}^{(q)}(x) +$$

$$+ \sum_{j=0}^{1} \sum_{i=0}^{2m} \frac{f^{(r)}(\xi_{v,i,j}) - f^{(r)}(x)}{(r-j)!} (t_{v,i}-t)^{r-j} u_{v,i,j}^{(q)}(x)$$

Noting (3), (4) and Lemma 3, we have

$$q!\, T_{\Delta n}^{(q)}(f;x) = q!\, f^{(q)}(x) + \sum_{p=0}^{r} f^{(p)}(x) \left[\frac{d^q}{dt^q} R_{p,m}(t,x) \right]_{t=x}$$

$$O(h_v^{r-q}\omega(f^{(r)};h_v)) .$$

From (5), we obtain

$$\left[\frac{d^q}{dt^q} R_{p,m}(x,t) \right]_{t=x} = \sum_{j=0}^{1} \sum_{i=0}^{2m} \left[\sum_{l=2m+3}^{\infty} c_{p,l}(t-x)^l \right]_{t=t_{v,j}}^{(j)} u_{v,i,j}^{(q)}(x) = o(h_v^{2m+3-q}) .$$

Hence,

$$\left| T_{\Delta n}^{(q)}(f;x) - f^{(q)}(x) \right| \leqslant M_1 h_v^{r-q} \omega(f^{(r)};h) + M_2 h_v^{2m+3-q} \left(\sum_{p=0}^{r} \| f^{(p)} \| \right) .$$

The proof of lemma is completed.

It is easy to verify

$$u''_{v,2m,1}(x_v+) = 2a \prod_{i=1}^{2m} \left(2\sin\frac{x_v - t_{v,i}}{2} \right) = \left[\sin\frac{x_{v+1}-x_v}{2} \right]^{-1},$$

$$u''_{v,2m,1}(x_{v+1}-) = 2a \left[\prod_{i=1}^{2m-1} \left(2\sin\frac{x-t_{v,i}}{2} \right) \left(2\sin\frac{x-x_v}{2} \right)^2 \right]'_{x=x_{v+1}}$$

$$= 2\cos\frac{x_{v+1}-x_v}{2} \Big/ \sin\frac{x_{v+1}-x_v}{2} + 2\sum_{j=1}^{2m+1} \cos\frac{x_{v+1}-t_{v,j}}{2} \Big/ \sin\frac{x_{v+1}-t_{v,j}}{2} \tag{6}$$

$$u''_{v,0,1}(x_{v+1}-) = \left[\sin\frac{x_v - x_{v+1}}{2} \right]^{-1},$$

$$u''_{v,0,1}(x_v+) = 2\cos\frac{x_v - x_{v+1}}{2} \Big/ \sin\frac{x_v - x_{v+1}}{2}$$

$$+ 2\sum_{j=1}^{2m-1} \cos\frac{x_v - t_{v,j}}{2} \Big/ \sin\frac{x_v - t_{v,j}}{2}$$

The C^2-continuity of $T_{\Delta n}^*(f;x)$ gives equations

$$T_{\Delta_n}^*(x_{v-1}) u''_{v-1,0,1}(x_v-) + T_{\Delta_n}^{*\prime}(x_v)[u''_{v-1,2m,1}(x_v-) - u''_{v,0,1}(x_v+)] - T_{\Delta_n}^{*\prime}(x_{v+1}) u''_{v,2m,1}(x_v+)$$

$$= \sum_{i=0}^{2m} f(t_{v,i}) u''_{v,i,0}(x_v+) - \sum_{i=0}^{2m} f(t_{v-1,i}) u''_{v-1,i,0}(x_v-) \quad (v=0,1,\cdots,n-1), \tag{7}$$

and (6) shows that it is a system of equations with dominant diagonal, when $\|\Delta_n\| = \underset{v}{\mathrm{Max}}\, h_v$ is small. Thus, we obtain

Theorem 1 For sufficiently small $\|\Delta_n\|$, there exists an unique trigonometric spline satisfying all conditions in definition.

To estimate the rate of convergence, we rewrite equations (7) in the following form

$$\frac{u''_{\nu-1,0,1}(x_\nu-)}{u''_{\nu-1,2m,1}(x_\nu-)-u''_{\nu,0,1}(x_\nu+)}[T^{*\prime}_{\Delta_n}(x_{\nu-1})-f'(x_{\nu-1})]+[T^{*\prime}_{\Delta_n}(x_\nu)-f'(x_\nu)]$$

$$-\frac{u''_{\nu,2m,1}(x_\nu+)}{u_{\nu-1,2m,1}(x_\nu-)-u''_{\nu,0,1}(x_\nu+)}[T^{*\prime}_{\Delta_n}(x_{\nu+1})-f'(x_{\nu+1})]$$

$$=\frac{1}{u_{\nu-1,2m,1}(x_\nu-)-u''_{\nu,0,1}(x_\nu+)}\Big[\sum_{i=0}^{2m}f(t_{\nu,i})u''_{\nu,i,0}(x_\nu+)+f'(x_\nu)u''_{\nu,0,1}(x_\nu+)+$$

$$f'(x_{\nu+1})u''_{\nu,2m,1}(x_\nu+)\Big]-\frac{1}{u''_{\nu-1,2m,1}(x_\nu-)-u''_{\nu,0,1}(x_\nu+)}\Big[\sum_{i=0}^{2m}f(t_{\nu-1,i})u''_{\nu-1,i,0}(x_\nu-)$$

$$+f'(x_{\nu-1})u''_{\nu-1,0,1}(x_\nu-)+f'(x_\nu)u''_{\nu-1,2m,1}(x_\nu-)\Big]$$

$$=I_{\nu,1}-I_{\nu,2},\quad \nu=0,1,\cdots,n-1. \qquad (8)$$

When $r\geqslant 2$, lemma 4 shows

$$I_{\nu,1}=\frac{1}{u''_{\nu-1,2m,1}(x_\nu-)-u''_{\nu,0,1}(x_\nu+)}f''(x_\nu)+O(h_\nu^{2m+2})+O(h_\nu^{r-1}\omega(f^{(r)};h_\nu)),$$

$$I_{\nu,2}=\frac{1}{u''_{\nu-1,2m,1}(x_\nu-)-u''_{\nu,0,1}(x_\nu+)}f''(x_\nu)+O(h_{\nu-1}^{2m+2})+O(h_\nu^{r-1}\omega(f^{(r)};h_{\nu-1})).$$

Hence, the right-hand side of (8) is $O(\|\Delta_n\|^{2m+2})+O(\|\Delta_n\|^{r-1}\omega(f^{(r)};\|\Delta_n\|))$. When $r=1$, we can prove $I_{\nu,i}=O(\|\Delta_n\|^{2m+2})+O(\omega(f';\|\Delta_n\|))$, $i=1,2$, too, and so

$$I_{\nu,1}-I_{\nu,2}=O(\|\Delta_n\|^{2m+2})+O(\|\Delta_n\|^{r-1}\omega(f';\|\Delta_n\|)).$$

Because the system of equations has dominant diagonal, we get

$$\operatorname*{Max}_{\nu}|T^{*\prime}_{\Delta_n}(f;x_\nu)-f'(x_\nu)|=O(\|\Delta_n\|^{2m+2})+O(\|\Delta_n\|^{r-1}\omega(f^{(r)};\|\Delta_n\|)). \qquad (9)$$

Now, we are able to prove

Theorem 2 If $f^{(r)}(x)\in C[0,1]$ $(1\leqslant r\leqslant 2m+2)$, the for $q=0,1$,

$$\|T^{*(q)}_{\Delta_n}(x)-f^{(q)}(x)\|=O(\|\Delta_n\|^{2m+3-q})+O(\|\Delta_n\|^{r-q}\omega(f^{(r)};\|\Delta_n\|)). \qquad (10)$$

where $O(\|\Delta_n\|^{2m+3-q})$ depends on $\sum_{p=0}^{r}\|f^{(p)}\|$. When $\operatorname{Max}h_\mu/\operatorname{Min}h_\nu\leqslant\beta<\infty$, $q\leqslant r$,

$$\|T^{*(q)}_{\Delta_n}(f;x)-f^{(q)}(x)\|=O(\|\Delta_n\|^{2m+3-q}+\|\Delta_n\|^{r-q}\omega(f^{(r)};\|\Delta_n\|)).$$

Proof Applying lemma 3, 4 and (9) to the following equations

$$T^{*(q)}_{\Delta_n}(f;x)-f^{(q)}(x)=T^{*(q)}_{\Delta_n}(f;x)-T^{(q)}_{\Delta_n}(f;x)+T^{(q)}_{\Delta_n}(f;x)-f^{(q)}(x)$$

$$=[T^{*\prime}_{\Delta_n}(x_\nu)-f'(x_\nu)]u^{(q)}_{\nu,0,1}(x)+[T^{*}_{\Delta_n}(x_{\nu+1})-f'(x_{\nu+1})]u^{(q)}_{\nu,2m,1}(x)+T^{(q)}_{\Delta_n}(f;x)-f^{(q)}(x),$$

we can easily complete the proof of the theorem.

Theorem 3 Suppose $f(x)\in C[0,1]$ is a periodic function with period 1, $\operatorname{Max}h_\nu/\operatorname{Min}h_\mu\leqslant\beta<\infty$, then

$$\|T^{*}_{\Delta_n}(f;x)-f(x)\|=O(\omega(f;\|\Delta_n\|)).$$

Proof By similar method used in the proof of previous theorem, we can prove

$$\operatorname*{Max}_{\nu} |T^{*\prime}_{\Delta_n}(x_\nu)| = O(\|\Delta_n\|^{-1}\omega(f;\ \|\Delta_n\|)).$$

Applying lemma 3, We have, for $x \in [x_\nu, x_{\nu+1}]$,

$$T^{*}_{\Delta_n}(f; x) - f(x) = \sum_{i=0}^{2m} [f(t_{\nu,i}) - f(x)]u_{\nu,i,0}(x)$$
$$+T^{*\prime}_{\Delta_n}(x_\nu)u_{\nu,0,1}(x) + T^{*\prime}_{\Delta_n}(x_{\nu+1})u_{\nu,2m,1}(x)$$
$$= O(\omega(f;\ \|\Delta_n\|)).$$

In the sequel, We discuss another kind of trigonometric splines. Because the process of proof is similar (not the same), and so the details are omitted.

Suppose $S_{\Delta_n}(f; x)$ satisfies

(i) In each $[x_\nu, x_{\nu+1}]$, $\nu = 0, 1, \cdots, n-1$, $S_{\Delta_n}(f; x)$ is a trigonometric polynomial of degree $\leqslant m$,

(ii) $S_{\Delta_n}(f; x) \in C^1[0, 1]$,

(iii) $S_{\Delta_n}(f; t_{\nu,i}) = f(t_{\nu,i})$, $\nu = 0, 1, \cdots, n-1$, $i = 1, 2, \cdots, 2m-1$,

(iv) $S_{\Delta_n}(f; 0) = f(0)$, $S_{\Delta_n}(f; 1) = f(1)$.

Theorem 4 There exists an unique trigonometric spline satisfying previous conditions for sufficiently small $\|\Delta_n\|$.

Theorem 5 Suppose $f(x) \in C^r[0,1]$ $(0 \leqslant r \leqslant 2m)$, then

$$\|S_{\Delta_n}(f; x) - f(x)\| = O(\|\Delta_n\|^r\omega(f^{(r)};\ \|\Delta_n\|) + \|\Delta_n\|^{2m+1-q}\sum_{p=0}^{r}\|f^{(p)}\|).$$

If $\operatorname{Max} h_\nu / \operatorname{Min} h_\mu \leqslant \beta < \infty$, then

$$\|S^{(q)}_{\Delta_n}(f; x) - f^{(q)}(x)\| = O(\|\Delta_n\|^{r-q}\omega(f^{(r)};\ \|\Delta_n\|) +$$
$$+ \|\Delta_n\|^{2m+1-q}\sum_{p=0}^{r}\|f^{(p)}\|) \quad (q < 2m).$$

Using lemma 1 and the method frequently used in the approximation theory, we can give some inverse theorems. For example, it can be proved that $\|T^{*}_{\Delta_n}(f; x) - f(x)\| = O(n^{-a})$ $(0 < a < 1)$ implies $f(x) \in \operatorname{Lip} a$, where the partition is determined by $x_\nu = \frac{\nu}{n}$, $\nu = 0, 1, \cdots, n$.

Remark I The second term in the right-hand side of (10) can not be removed. It can be demonstrated by function $f(x) = x$, for which, the first term vanishes for $r > 1$, but $T^{*}_{\Delta_n}(x; t) \not\equiv t$.

Remark 2 In the theory of polynomial spline, there are beautiful inverse theorems, and saturation problem can be solved easily. But for the trigonometric spline, the situation is quite different. Especially, the saturation problem is not easy to solve.

Remark 3 In[3], Wuong Zu-yin discussed interpolating trigonometric spline of degree 1, which is a special case of the theorem in this paper. Furthermore, our results are much better than those in[3].

References

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follows:

1. Take real initial vector $\|v_1\|_2 = 1$, $u \leftarrow Av_1$

2. For $j = 2, \cdots, k$ do i) $\tilde{\gamma}_j \leftarrow \sqrt{u^T u}$, if $\tilde{\gamma}_j = 0$ then stop, ii) $v_j \leftarrow u/\tilde{\gamma}_j$, iii) $u \leftarrow Av_j + \tilde{\gamma}_j v_{j-1}$.

Now let G be a B-skew symmetric matrix[2], then iG is a B-Hermitian m matrix. The three term recurrence relation associated with iG is

$$\gamma_{j+1} v_{j+1} = iG v_j - \alpha_j v_j - \beta_j v_{j-1}, \; j = 1, 2, \cdots, \; \beta_1 = 0, \tag{6}$$

where

$$\alpha_j = (iG v_j, v_j)_B, \beta_j = (iG v_j, v_{j-1})_B, \; \gamma_j = (iG v_{j-1}, v_j)_B. \tag{7}$$

Taking v_1 a real vector, we can make all $v_i, i = 2, 3, \cdots$, real vectors and we also have

$$\alpha_j = 0, \; \beta_{j+1} = i\,(G v_{j+1}, v_j)_B = -\gamma_{j+1}, j = 1, 2, \cdots,$$

Therefore, if we let $\gamma_j = i\tilde{\gamma}_j$, then we get rather simple recurrence relations

$$\tilde{\gamma}_{j+1} v_{j+1} = G v_j + \tilde{\gamma}_j v_{j-1}, \; j = 1, 2, \cdots,$$

Usually we take

$$\tilde{\gamma}_{j+1} = \|G v_j + \tilde{\gamma}_j v_{j-1}\|_B, \; j = 1, 2, \cdots. \tag{8}$$

Thus we get the same skew symmetric tridiagonal matiix T_k in (5) provided that $\tilde{\gamma}_{j+1}, j = 1, 2, \cdots$, are computed in (8). The Lanczos algorithm for the B-skew symmetric matrix G is as follows:

1. Take a real initial vector v_1, $\|v_1\|_B = 1$, $u \leftarrow A v_1$

2. For $j = 2, 3, \cdots, k$ do i) $\tilde{\gamma}_j \leftarrow \sqrt{u^T B u}$. if $\tilde{\gamma}_j = 0$ then stop, ii) $v_j \leftarrow u/\tilde{\gamma}_j$ iii) $u \leftarrow Av_j + \tilde{\gamma}_j v_{j-1}$.

References

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