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Asymptotic Estimation for Hermite-Fejér Type Interpolation of Higher Order*

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I. Introduction

Let $f \in C[-1,1]$ and $x_k = x_{kn} = \cos\theta_k = \cos(2k-1)\pi/(2n)$ $(k=1,\cdots,n)$ be the zeros of the Chebyshev polynomial $T_n(x) = \cos n\theta$ $(x = \cos\theta)$. Let $\omega(t)$ be a given modulus of continuity and $H_\omega = \{f; \omega(f,t) \leq \omega(t),$ for all $t > 0\}$. In this paper, c will always denote different constant independent of x, n and f and the sign "$A \sim B$" means that there exist two positive constants $c_1 < c_2$ independent of n, x and f such that $c_1 A \leq B \leq c_2 A$.

Consider the following interpolation polynomial of degree $\leq 4n-1$ such that

$R_{4n-1}(f, x_k) = f(x_k)$ $(k=1,\cdots,n)$, $R^{(j)}_{4n-1}(f, x_k) = 0$ $(j=1,2,3; k=1,\cdots,n)$,

that is given by

$$R_{4n-1}(f,x) = \sum_{k=1}^{n} f(x_k) S_k(x),$$

where

$$S_k(x) = F_k(x) + G_k(x) + H_k(x),$$

$$F_k(x) = \frac{1}{2n^4}(1-x^2)\cdot(1-x_k^2)(T_n(x)/(x-x_k))^4,$$

$$G_k(x) = \frac{4n-1}{6n^4}(x-x_k)^2\cdot(1-xx_k)(T_n(x)/(x-x_k))^4,$$

$$H_k(x) = \frac{1}{2n^4}(1-xx_k)^2(T_n(x)/(x-x_k))^4.$$

The polynomial $R_{4n-1}(f,x)$ was first introduced by Krylov and Steuermann[1], later, it was further studied by many authors [2—9,11]. Here we state some main results.

Theorem A[5]. The following estimate is valid

$$\sup_{f \in H_\omega} \| R_{4n-1}(f) - f \| \sim \frac{1}{n} \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right), \tag{1.1}$$

here $\|\cdot\|$ denotes the usual supremum norm.

Theorem B[9]. *If* $-1 < x < 1$, then

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$$|f(x)-R_{4n-1}(f,x)|\leqslant c\{\omega\left(f,\frac{|T_n(x)|}{n}\right)+\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(f,\frac{\sqrt{1-x^2}}{k}+\frac{1}{k}\right)\}.\quad(1.2)$$

Theorem C [1, Coro. 2, 3]. If $f\in C[-1,1]$, $x=\cos\theta(0\leqslant\theta\leqslant\pi)$ and $|\theta_{k_0}-\theta|\leqslant\frac{\pi}{2n}$, then

$$|f(x)-R_{4n-1}(f,x)|\leqslant c\{\omega(f,\ |\theta-\theta_{k_0}|\sqrt{1-x^2}+|\theta-\theta_{k_0}|^2)+\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(f,\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\}\quad(1,3)$$

From the above ones it is natural to ask what is the exact pointwise degree of approximation. The first purpose of this paper is to prove the degree in (1,3) is exact in fact.

For $f\in$ Lip1, S. J. Goodenough and T. M. Mills further proved the following.

Theorem D[8]. If $-1\leqslant x\leqslant 1$, then

$$\sup_{f\in\mathrm{Lip}1}|f(x)-R_{4n-1}(f,x)|=\frac{4}{3\pi}T_n^4(x)\sqrt{1-x^2}\frac{\ln n}{n}+O(1/n)\quad(n\to\infty).\qquad(1,4)$$

It should be noted that estimate (1.4) is not uniform in $-1\leqslant x\leqslant 1$. Therefore, the second aim of the present paper is to give an uniformly asymptotic estimate.

2. the Main Results

Our main results are the following

Theorem 2.1 The following estimate is valid

$$\sup_{f\in H_\omega}|f(x)-R_{4n-1}(f,x)|\sim\{\omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right)+\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\}.\qquad(2.1)$$

Theorem 2.2 *If* $-1<x<1$, then the following is valid uniformly:

$$\sup_{f\in\mathrm{Lip}1}|f(x)-R_{4n-1}(f,x)|=\frac{T_n^4(x)}{n}\{\frac{4}{3\pi}\sqrt{1-x^2}\ln(1+n\arccos|x|)+\frac{2}{3}|x|+\frac{(4n^2-1)(1-x^2)}{6n^3|x-x_j|}+\frac{(1-x^2)(1-x_j^2)+(1-xx_j)^2}{2n^3|x-x_j|^3}+O\left(\sqrt{1-x_j^2}+\frac{1}{n}\right)\}\qquad(2.2)$$

where x_j is the nearest zero to x.

3. Proofs of the Theorems

Proof of Theorem 2,1. Let $|k-j|=i$, $(k\neq j)$. The following inequalities are obvious:

$$\sin\theta\leqslant 2\sin\frac{1}{2}(\theta+\theta_k),\qquad(3.1)$$

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$$\sin\theta_k \leqslant 2\sin\frac{1}{2}(\theta+\theta_k), \tag{3.2}$$

$$\sin\frac{1}{2}(\theta-\theta_k)| \leqslant \sin\frac{1}{2}(\theta+\theta_k) \tag{3.3}$$

$$i\pi/(2n) \leqslant |\theta-\theta_k| \leqslant 2i\pi/n.\quad (k\neq j) \tag{3.4}$$

Using (3.1), (3.3) and (3.4), it is easy to verify that

$$|x-x_k| \sim (i\sqrt{1-x^2}/n + i^2/n^2)\quad (k\neq j). \tag{3.5}$$

Noticing the fact

$$\frac{n|\sin\frac{1}{2}(\theta-\theta_j)|}{|T_n(x)|} = \frac{n|\sin\frac{1}{2}(\theta-\theta_j)|}{|\sin n(\theta-\theta_j)|} \sim 1, \tag{3.6}$$

we have obviously that

$$|x-x_j| \sim (|T_n(x)|\sqrt{1-x^2}/n + T_n^2(x)/n^2). \tag{3.7}$$

Since $R_{4n-1}(f,x)$ is a positive linear operator, we have

$$\sup_{f\in H_\omega}|f(x)-R_{4n-1}(f,x)| = \frac{(1-x^2)T_n^4(x)}{2n^4}\sum_{k=1}^{n}\omega(|x-x_k|)\frac{1-x_k^2}{(x-x_k)^4} +$$
$$+\frac{(4n^2-1)T_n^2(x)}{6n^2}\sum_{k=1}^{n}\omega(|x-x_k|)(1-xx_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2 +$$
$$+\frac{T_n^4(x)}{2n^4}\sum_{k=1}^{n}\omega(|x-x_k|)\frac{(1-xx_k)^2}{(x-x_k)^4} \triangleq R_1(x)+R_2(x)+R_3(x) \tag{3.8}$$

First, we estimate $R_2(x)$. Write

$$R_2(x) = \frac{(4n^2-1)T_n^2(x)}{6n^2}\left\{\omega(|x-x_j|)\frac{T_n^2(x)(1-xx_j)}{n^2(x-x_j)^2} + \right.$$
$$\left. + T_n^2(x)\sum_{\substack{k=1\\k\neq j}}^{n}\omega(|x-x_k|)\frac{(1-xx_k)}{n^2(x-x_k)^2}\right\} = \frac{(4n^2-1)T_n^2(x)}{6n^2}\left\{R_{21}(x)+R_{22}(x)\right\} \tag{3.9}$$

Since

$$\frac{(1-xx_k)}{(x-x_k)^2} = \frac{\sin^2\frac{1}{2}(\theta+\theta_k)+\sin^2\frac{1}{2}(\theta-\theta_k)}{4\sin^2\frac{1}{2}(\theta+\theta_k)\sin^2\frac{1}{2}(\theta-\theta_k)} \sim \frac{1}{\sin^2\frac{1}{2}(\theta-\theta_k)}\quad (k=1,\cdots,n), \tag{3.10}$$

it follows from (3.4)—(3.7) that

$$R_{21}(x) \sim \omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2} + \frac{T_n^2(x)}{n^2}\right), \tag{3.11}$$

and

$$R_{22}(x) \sim \frac{T_n^2(x)}{n}\sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right) \tag{3.12}$$

Combining (3.9), (3.11) and (3.12), we obtain

$$R_2(x) \sim \left\{T_n^2(x)\,\omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right) + \frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right\} \tag{3.13}$$

Similarly, we have

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$$R_3(x)\sim\left\{\omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right)+T_n^4(x)\sum_{k=1}^{n}\omega\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^4}\right)\frac{1}{k^4}\right\}\sim$$

$$\sim\omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right). \tag{3.14}$$

From (3.13) and (3.14), we get

$$\{R_2(x)+R_3(x)\}\sim\left\{\omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right)+\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(\frac{\sqrt{-x^2}}{k}+\frac{1}{k^2}\right)\right\} \tag{3.15}$$

Using (3.1), (3.2), (3.4) and (3.6), the following holds

$$R_1(x)<c\left\{\omega\left(\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right)+\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right\}. \tag{3.16}$$

Finally, combining (3.8), (3.15) and (3.16), we obtain (1.2). The proof is completed.

It is clear that (1.3) is a direct consequence of Theorem 2.1. In view of (3.6), (1.3) can be rewrited by

$$|f(x)-R_{4n-1}(f,x)|<c\left\{\omega\left(f,\frac{|T_n(x)|}{n}\sqrt{1-x^2}+\frac{T_n^2(x)}{n^2}\right)+\right.$$

$$\left.+\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(f,\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right\}. \tag{3.17}$$

Now we compare (3.17) with (1.2). Let $f\in \mathrm{Lip}_M\alpha$ $(0<\alpha<1)$ and $x^*=x_n^*=\cos\theta^*=\cos(\frac{1}{2n}-\frac{1}{n^2})\pi$, then $|T_n(x^*)|=\sin\pi/n\sim 1/n$. From (3.17) it follows that $|f(x^*)-R_{4n-1}(f,x^*)|<c\left\{\omega(f,\frac{1}{n^3})+\frac{1}{n^5}\sum_{k=1}^{n}\omega\left(f,\frac{1}{nk}+\frac{1}{k^2}\right)\right\}<\frac{cM}{n^{3\alpha}}$, and from (1.2), $|f(x^*)-R_{4n-1}(f,x^*)|<cM/n^{2\alpha}$. From the above it follows that estimate (1.3) is slightly better than (1.2).

Proof of Theorem 2.2. From (3.8) we have

$$\sup_{f\in \mathrm{Lip}1}|f(x)-R_{4n-1}(f,x)|=\frac{(1-x^2)T_n^4(x)}{2n^4}\sum_{k=1}^{n}\frac{(1-x_k^2)}{|x-x_k|^3}+$$

$$+\frac{(4n^2-1)T_n^2(x)}{6n^2}\sup_{f\in \mathrm{Lip}1}|f(x)-H_{2n-1}(f,x)|+\frac{T_n^4(x)}{2n^4}\sum_{k=1}^{n}\frac{(1-xx_k)^2}{|x-x_k|^3}$$

$$=\Delta_1(x)+\Delta_2(x)+\Delta_3(x), \tag{3.18}$$

where $H_{2n-1}(f,x)$ is the Hermite-Fejér operator based on the zeros of $T_n(x)$. By Xie Tingfan[10],

$$\sup_{f\in \mathrm{Lip}1}|f(x)-H_{2n-1}(f,x)|=\frac{T_n^2(x)}{n}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln(1+n\arccos|x|)+|x|+\right.$$

$$\left.+\frac{1-x^2}{n|x-x_j|}+O(\sqrt{1-x^2})\right\},$$

from where it follows that

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$$\Delta_2(x)=\frac{T_n^4(x)}{n}\left\{\frac{4}{3\pi}\sqrt{1-x^2}\ln(1+n\arccos|x|)+\frac{2}{3}|x|+\right.$$

$$\left.+\frac{2(1-x^2)}{3n|x-x_j|}-\frac{1-x^2}{6n^3|x-x_j|}+O\left(\sqrt{1-x^2}+\frac{1}{n}\right)\right\}. \tag{3.19}$$

By (3.1) and (3.2) we get

$$\Delta_1(x)=\frac{T_n^4(x)}{2n^4}\left\{\frac{(1-x^2)(1-x_j^2)}{|x-x_j|^3}+\sum_{k\neq j}\frac{\sin^2\theta\sin^2\theta_k}{2^3|\sin^3\frac{1}{2}(\theta-\theta_k)\sin^3\frac{1}{2}(\theta+\theta_k)|}\right\}=$$

$$=\frac{T_n^4(x)}{n}\left\{\frac{(1-x^2)(1-x_j^2)}{2n^3|x-x_j|^3}+O(\sqrt{1-x^2})\right\}. \tag{3.20}$$

For the estimation of $\Delta_3(x)$, it is easy to see that

$$\Delta_3(x)=\frac{T_n^4(x)}{n}\left\{\frac{(1-xx_j)^2}{2n^3|x-x_j|^3}+O\left(\sqrt{1-x^2}+\frac{1}{n}\right)\right\}. \tag{3.21}$$

Finally, combining (3.18)-(3.21), (2.2) follows. The proof is completed.

If $x\in(-1,1)$ is fixed, then $\ln(1+n\arccos|x|)=\ln n+O(1)$, hence

$$\sup_{f\in \mathrm{Lip}1}|f(x)-R_{4n-1}(f,x)|=\frac{4}{3\pi}T_n^4(x)\sqrt{1-x^2}\frac{\ln n}{n}+O(|T_n(x)|/n)\quad(-1<x<1).$$

Then we have the following

Corollary. Suppose $x\in[-1,1]$ is fixed, then

$$\sup_{f\in \mathrm{Lip}1}|f(x)-R_{4n-1}(f,x)|=\begin{cases}\frac{2}{3n}+O\left(\frac{1}{n^2}\right), & \text{as } |x|=1,\\ \frac{4}{3\pi}T_n^4(x)\sqrt{1-x^2}\frac{\ln n}{n}+O\left(\frac{|T_n(x)|}{n}\right) & \text{as } -1<x<1,\end{cases}$$

where the sign "O" depends only on x.

Estimate (3.23) improves Theorem D.

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