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The Uniqueness and Approximate Solution of the Positive Solution of the Uniform u_0-convex Operator Equation*

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M.A.Krasnosel'skii [1] proposed that "under what conditions does the solution of convex operator equation Ax=x exist and unique?". Because problem itself is more difficult, its development is quite slow. Lately Guo Da-Jun [2] suggested explicitly that "under what conditions does u_0-convex operator have unique positive fixed point?".

Chang and Kang [3] studied conditions under which the uniform u_0-convex operator equations have positive solution.

In this paper we discuss uniqueness of positive solution of the uniform u_0-convex operator equation and construct the solution by method of successive approximation.

Let E be real Banach space, P be a normal cone in E, $u_0 \in P$ and $E_{u_0}=\{x \mid x\in E,$ there exist $\lambda>0$ such that $-\lambda u_0 \leqslant x \leqslant \lambda u_0\}$ and $P_{u_0}=P\cap E_{u_0}$. If we set for $x\in E_{u_0}$, $\|x\|_{u_0}=\inf\{\lambda \mid \lambda>0,\ -\lambda u_0\leqslant x\leqslant \lambda u_0\}$, the nonlinear operator $A:P_{u_0}\to P_{u_0}$ is called monotonic increasing under CH_{u_0}-meaning, if the operator

$$Bx=\begin{cases}\|x\|_{u_0}^2 A\left(\dfrac{x}{\|x\|_{u_0}^2}\right) & x\neq\theta\\ \theta & x=\theta\end{cases}$$

is monotonic increasing (i, e $\theta\leqslant x\leqslant y\to Bx\leqslant By$).

Other notations are same as [1] and [4].

The result of this paper is following:

Theorem. Let E be a real Banach space. Let P be a normal cone in E. Suppose that the operator $A:P_{u_0}\to P_{u_0}$ is uniform u_0-convex and monotonic increasing under CH_{u_0}-meaning. If there exist $v_0, w_0>\theta$ in P_{u_0} such that

$$\frac{v_0}{\|v_0\|_{u_0}^2}\leqslant A\left(\frac{v_0}{\|v_0\|_{u_0}^2}\right),\quad A\left(\frac{w_0}{\|w_0\|_{u_0}^2}\right)\leqslant\frac{w_0}{\|w_0\|_{u_0}^2} \qquad (1)$$

Then operator equation $Ax=x$ has unique nonzero positive solution $x^*\in P_{u_0}$ and for an arbitrary $y_0\in P_{u_0}$ $(y>\theta)$ we construct sequences by successive approximation

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$$y_n = By_{n-1} \text{ and } x_n = A\left(\frac{y_{n-1}}{\|y_{n-1}\|_{u0}^2}\right) \quad (n = 1, 2, 3, \cdots\cdots). \tag{2}$$

Then $\|x_n - x^*\| \to 0\ (n \to \infty)$, i, e x_n is the approximate solution of the operator equation $Ax = x$.

Proof: By assumption of this theorem, operator $A: P_{u0} — P_{u0}$ is uniform u_0 -convex and monotonic increasing under CH_{u0}-meaning. By theorem 2.1 of [3], we know that operator

$$Bx = \begin{cases} \|x\|_{u0}^2 A\left(\dfrac{x}{\|x\|_{u0}^2}\right), & x \neq \theta \\ \theta & x = \theta \end{cases} \quad (x \in P_{u0})$$

maps P_{u0} into P_{u0} and is uniform u_0-concave and monotonic increasing. By (1) we obtain $v_0 < Bv_0$, $Bw_0 \leqslant w_0$.

From theorem 1 of [4], operator equation $By = y$ has nonzero solution $y^* = By^*$.

By theorem 6.3 of [1], we know that y^* is unique solution. From definition of operator B, the operator equation $Ax = x$ has positive solution $x^* = \dfrac{y^*}{\|y^*\|_{u0}^2}$, i, e

$$A\left(\frac{y^*}{\|y^*\|_{u0}^2}\right) = \frac{y^*}{\|y^*\|_{u0}^2} \quad \text{or} \quad Ax^* = x^*.$$

By virtue of rotation between A and B, the existence and uniqueness of nonzero positive solution of operator equation $Ax = x$ and $By = y$ are equivalent. Hence operator equation $Ax = x$ has an unique nonzero positive solution.

Let y_0 be an arbitrary nonzero element of P_{u0}. We construct sequence of successive approximation

$$y_n = By_{n-1} \quad (n = 1, 2, 3, \cdots\cdots) \tag{3}$$

By theorem 6.7 of [1], $\|y_n - y^*\| \to 0\ (n \to \infty)$. Sinc the cone P is normal, $\|y_n - y^*\| \to 0\ (n \to \infty)$, i, e y_n are aproximate positive solutions of equation $By = y$. By (3) we obtain $\dfrac{y_n}{\|y_{n-1}\|_{u0}^2} = A\left(\dfrac{y_{n-1}}{\|y_{n-1}\|_{u0}^2}\right)$. Set $x_n = A\left(\dfrac{y_{n-1}}{\|y_{n-1}\|_{u0}^2}\right)$,

$$\|x_n - x^*\| = \left\|A\left(\frac{y_{n=1}}{\|_{n-1}\|_{u0}^2}\right) - \frac{y^*}{\|y^*\|_{u0}^2}\right\| = \left\|\frac{y_n}{\|y_{n-1}\|_{u0}^2} - \frac{y^*}{\|y^*\|_{u0}^2}\right\|$$

$$= \frac{1}{\|y_{n-1}\|^2 \|y^*\|_{u0}^2} \left\| y_n \|y^*\|_{u0}^2 - y^* \|y_{n-1}\|_{u0}^2 \right\|$$

$$= \frac{1}{\|y_{n-1}\|_{u0}^2} \left\| y_n \|y^*\|_{u0}^2 - y^* \|y^*\|_{u0}^2 + y^* \|y^*\|_{u0}^2 - y^* \|y_{n-1}\|_{u0}^2 \right\|$$

$$\leqslant \frac{1}{\|y_{n-1}\|_{u0}^2 \|y^*\|_{u0}^2} \left\{ \|y_n - y^*\| \cdot \|y^*\|_{u0}^2 + \|y^*\| \cdot \left| \|y^*\|_{u0}^2 - \|y_{n-1}\|_{u0}^2 \right| \right\}$$

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$$= \frac{1}{\|y_{n-1}\|_{u_0}^2 \|y^*\|_{u_0}^2} \{ \|y_n - y^*\| \cdot \|y^*\|_{u_0}^2 + \|y^*\| \cdot \left| \|y^*\|_{u_0} - \|y_{n-1}\| \right| \cdot \left| \|y^*\|_{u_0} + \right.$$

$$\left. + \|y_{n-1}\| \right| \} .$$

Since $\|y_n - y^*\| \to 0$ and $| \|y^*\|_{u_0} - \|y_{n-1}\|_{u_0} | \to 0$ (by continuity of u_0-norm), we obtain $\|x_n - x^*\| \to 0$ $(n \to \infty)$. Then

$x_n = A(\frac{y_{n-1}}{\|y_{n-1}\|_{u_0}^2})$ is the aproximate positive solution, where y_0 is an arbitrary nonzero element in P_{u_0}, $y_n = By_{n-1}$ $(n = 1, 2, 3, \cdots)$.

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