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On Subharmonic Extensions*

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Recently V. Anandam [1] has established the following theorem:

Suppose that $u(x)$ is a subharmonic function defined outside a compact set in R^n. Then for any sufficiently large $r>0$ there exist a nonconstant subharmonic function $s(x)$ in R^n and a constant $a\geqslant 0$ such that for $|x|>r$

$$u(x)=\begin{cases} s(x)+a\log\frac{1}{|x|}, & n=2,\\ s(x)+a|x|^{2-n}, & n>2. \end{cases} \tag{1}$$

The representation (1) is interesting for its simple appearance and gives new versions to some previous results on the behaviour of a subharmonic function near the point at infinity. But as a theorem on subharmonic extensions it appears unnecessarily restrictive. The purpose of the present paper is to improve it in this respect.

It is convinient to remind at first that any neighborhood of a compact set K in R^n has only a finite number of connected components intersecting K. Our result can now be stated as follows.

Theorem 1 Suppose that $u(x)$ is a subharmonic function defined outside a compact set K in R^n, and V is an arbitrary neighborhood of K. let $V_1, V_2, \cdots, V_m$ be all the connected components of V which intersect K, and x_i is an arbitrary point in V_i $(i=1,2,\cdots,m)$. Then there exist a nonconstant subharmonic function $s(x)$ in R_n and a constant $a\geqslant 0$ such that

$$u(x)=\begin{cases} s(x)+a\sum_{i=1}^{m}\log\frac{2}{|x-x_i|}, & n=2;\\ s(x)+a\sum_{i=1}^{m}|x-x_i|^{2-n}, & n>2 \end{cases} \tag{2}$$

outside $\overline{V}$.

Proof Suppose that V^* is a neighbourhood of K and $\overline{V^*}\subset V$. without loss of generality we may assume that $V=\bigcup_{i=1}^{m}V_i$ and $\overline{V}$ is compact and possesses no irregular boundary point, and that $u(x)$ is finite continuous in a neighbourhood of ∂V [2]. Denote by $D_V\varphi$ the Dirichlet solution in V with continuous boundary value

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φ. write

$$w(x)=\begin{cases}\sum_{i=1}^{m}\log|x-x_i|, & n=2;\\ \sum_{i=1}^{m}|x-x_i|^{2-n}, & n>2.\end{cases}$$

Since w is a nonharmonic and subharmonic function in V, we have

$$w(x)<(D_V w)(x), \qquad x\in V,$$

so that $D_V w - w$ has a positive minimum on ∂V^*. Hence for sufficiently large number $a\geqslant 0$ we have

$$(D_V w-w)\geqslant u-D_V u \qquad \text{on } \partial V^*$$

Therefore for all $x\in\partial V^*$

$$u+aw\leqslant D_V\,(u+aw). \tag{3}$$

The last inequality (3) holds for all $x\in\partial V$ (actually it becomes an equality for all $x\in\partial V$).

Therefore (3) holds for all $x\in V\sim V^*$.

Define

$$s(x)=\begin{cases}u(x)+aw(x), & x\in C(V);\\ D_V(u+aw), & x\in V.\end{cases}$$

We shall show that $s(x)$ satisfies the requirements stated in our theorem. Evidently it needs only to verify the subharmonicity of $s(x)$ for $x\in\partial V$.

For any $x_0\in\partial V$, denote by $C(x_0,r)$ the sphere (for $n>2$) or the circle (for $n=2$) with radius r and center x_0. Then for sufficiently small r, we have by (3)

$$s(x_0)=u(x_0)+aw(x_0)\leqslant\frac{1}{\Omega_n(r)}\int_{C(x_0,r)}\{u(x)+aw(x)\}d\Omega\leqslant\frac{1}{\Omega_n(r)}\int_{C(x_0,r)}S(x)d\Omega,$$

where $\Omega_n(r)$ denotes the area of the sphere (when $n>2$) or the length of the circle of radius r in R^n. The subharmonicity of $s(x)$ at $x=x_0$ follows.

Moreover $w(x)$ can always be chosen a nonconstant function since a is an arbitrarily large positive number.

Remark. It is easily seen that Anandams theorem coincides with the particurlar case of ours when V is a ball $(n>2)$ or a disk centred at $x=0$. The following example shows that in general the right side of the representation (2) should be expressed in terms of m distinct points x_i $(i=1,2,\cdots,$ m) instead of a single one.

Example. Let $k=\{0,x_1\}$, $u(x)=\log|x|-\log|x-x_1|$. $V=V_0\cup V_1=\{|x|<r\}\cup\{|x-x_1|<r\}$, where r is so small that $V_0\cap V_1=\phi$, Then there exists no nonconstant subharmonic function $s(x)$ in R^2 and a constant $a\geqslant 0$ such that

$$u(x)=s(x)-a\log|x|$$

outside V.

Proof. Suppose on the contrary that there exist a nonconstant subharmonic function $s(x)$ and a constant $a \geqslant 0$ such that $u(x) = s(x) - \log|x|$ outside V, then we have

$$s(x) = (1+a)\log|x| - \log|x - x_1|, \quad x \in \mathrm{R}^2 \sim \mathrm{V}.$$

For any $x_0 \in \partial \mathrm{V}_1$, denote by $\mathrm{C}(x_0, \rho)$ the circle with radius ρ and center x_0, write $\mathrm{C}' = \mathrm{C}(x_0, \rho) \cap \mathrm{V}_1$, $\mathrm{C}'' = \mathrm{C}(x_0, \rho) \cap \{\mathrm{R}^2 \sim \mathrm{V}_1\}$, $f_0(x) = \log|x|$, $f_1(x) = \log|x - x_1|$. Since $s(x)$ is a subharmonic fouction in R^2 and $(1+a)f_0 - f_1$ is a nonharmonic and superharmonic function in V_1, we have

$$s(x_0) \leqslant \frac{1}{2\pi}\int_{\mathrm{C}(x_0,\rho)} s(\rho e^{i\theta} + x_0)\mathrm{d}\theta = \frac{1}{2\pi}\int_{\mathrm{C}'} s(\rho e^{i\theta} + x_0)\mathrm{d}\theta +$$

$$+\frac{1}{2\pi}\int_{\mathrm{C}''} s(\rho e^{i\theta} + x_0)\mathrm{d}\theta \leqslant \frac{1}{2\pi}\int_{\mathrm{C}'} (D_{\mathrm{V}_1} s)(\rho e^{i\theta} + x_0)\mathrm{d}\theta + \frac{1}{2\pi}\int_{\mathrm{C}''} s(\rho e^{i\theta} + x_0)\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{\mathrm{C}'} \{D_{\mathrm{V}_1}[(1+a)f_0 - f_1]\}(\rho e^{i\theta} + x_0)\mathrm{d}\theta + \frac{1}{2\pi}\int_{\mathrm{C}''} s(\rho e^{i\theta} + x_0)\mathrm{d}\theta$$

$$< \frac{1}{2\pi}\int_{\mathrm{C}'} \{(1+a)f_0 - f_1\}(\rho e^{i\theta} + x_0)\mathrm{d}\theta + \frac{1}{2\pi}\int_{\mathrm{C}''} \{(1+a)f_0 - f_1\}(\rho e^{i\theta} + x_0)\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{\mathrm{C}(x_0,\rho)} \{(1+a)f_0 - f_1\}(\rho e^{i\theta} + x_0)\mathrm{d}\theta = (1+a)f_0(x_0) - f_1(x_0) = s(x_0).$$

We arrive therefore at a contradiction. Hence the assertion holds.

Theorem 1 can obviously be generalized as follows.

Theorem 2. Suppose that Ω is either a Green space [3] or an open Riemann surface of $\mathrm{O_G}$, $u(x)$ is a subharmonic function defined outside a compact set K in Ω and V is an arbitrary neighborhood of K. Let $\mathrm{V}_1, \mathrm{V}_2, \cdots, \mathrm{V}_m$ be all the connected components of V which intersect K.

i) For arbitrary $x_i \in \mathrm{V}_i$ $(i = 1, 2, \cdots, m)$ there exist a nonconstant subharmonic function $s(x)$ in Ω and a constant $a \geqslant 0$ such that

$$u(x) = s(x) + a\sum_{i=1}^{m} g(x, x_i), \qquad x \in \Omega \sim \mathrm{V},$$

where $g(x, x_i)$ $(i = 1, 2, \cdots, m)$ denotes the Green function or the modified Green function [4] with its unique singularity at $x = x_i$ respectively;

ii) For any function w superharmonic in Ω and nonharmonic in each V_i $(i = 1, 2, \cdots, m)$ there exist a nonconstant subharmcnic function $s(x)$ in Ω and a constant $a \geqslant 0$ snch that

$$u(x) = s(x) + a\mathrm{w}(x), \qquad x \in \Omega \sim \overline{\mathrm{V}}.$$

It is noted that the assertion ii) in Theorem 2 remains true when Ω is assumed to be a harmonic space [5] of more general type.

References

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