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Three Combinatorial Identities*

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Let

$$\langle x\rangle = 1 - q^{x}$$

and

$$\begin{bmatrix} x \\ k \end{bmatrix} = \prod_{i=1}^{k} \frac{\langle x-i+1\rangle}{\langle 1\rangle}.$$

If q is replaced by q^{-1}, the corresponding notations are denoted by $\langle x\rangle_{-}$ and $\begin{bmatrix} x \\ k \end{bmatrix}_{-}$ respectively.

Recently, in research on the enumeration of lattice paths, I have obtained the following combinatorial identities:

i. $$\sum_{\substack{0\le i<m\\ 0\le j<n}} \frac{\langle 1\rangle}{\langle i+j+1\rangle} \begin{bmatrix} i+j+1 \\ i \end{bmatrix} \begin{bmatrix} i+j+1 \\ j \end{bmatrix} \frac{\langle 1\rangle}{\langle m+n-i-j-1\rangle} \cdot \begin{bmatrix} m+n-i-j-1 \\ m-i \end{bmatrix} \begin{bmatrix} m+n-i-j-1 \\ n-j \end{bmatrix} \ast$$
$$\cdot\, q^{(2j+1)(m-i)+n-j-1} = q^{n}(1+q)\frac{\langle 1\rangle}{\langle m+n\rangle} \begin{bmatrix} m+n \\ m+1 \end{bmatrix} \begin{bmatrix} m+n \\ n+1 \end{bmatrix}.$$

ii. $$\sum_{0\le i\le j\ \min(n,k)} \frac{\langle j-i+1\rangle}{\langle k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} q^{n(2m-2k+i+j)} \ast$$
$$\frac{\langle j-i+1\rangle_{-}}{\langle m+n-k+1\rangle_{-}} \begin{bmatrix} m+n-k+1 \\ n-j \end{bmatrix}_{-} \begin{bmatrix} m+n-k+1 \\ n-i+1\ 1 \end{bmatrix}_{-}$$
$$= \frac{\langle 1\rangle}{\langle m+n+1\rangle} \begin{bmatrix} m+n+1 \\ m \end{bmatrix} \begin{bmatrix} m+n+1 \\ n \end{bmatrix}.$$

iii. $$\sum_{0\le i\le j\le n} \frac{\begin{vmatrix} \langle k+1\rangle & \langle i\rangle \\ q\langle k\rangle & \langle j+1\rangle \end{vmatrix}}{\langle k+1\rangle\langle j+1\rangle} \begin{bmatrix} k+i \\ i \end{bmatrix} \begin{bmatrix} k+j \\ j \end{bmatrix} q^{i+j+2n(m-k-1)} \ast$$
$$\cdot\, \frac{\begin{vmatrix} \langle m-k\rangle_{-} & \langle n-j\rangle_{-} \\ q\langle m-k-1\rangle_{-} & \langle n-i+1\rangle_{-} \end{vmatrix}}{\langle m-k\rangle_{-}\ \langle n-i+1\rangle_{-}} \begin{bmatrix} m+n-k-j-1 \\ n-j \end{bmatrix} \begin{bmatrix} m+n-k-i-1 \\ n-i \end{bmatrix}_{-}$$
$$= \frac{\langle 1\rangle}{\langle m+n+1\rangle} \begin{bmatrix} m+n+1 \\ m \end{bmatrix} \begin{bmatrix} m+n+1 \\ n \end{bmatrix}.$$

Taking the limit of $q\to 1$, we note that the above mentioned formulae become double convolution-type identities on binomial coefficients. Their direct proofs have not been known from author.

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