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TOPOLOGIES GENERATED BY FUNCTORS*

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1. This note aims to introduce the notion of the categories with topologies and to discuss topologies generated by functors. A topological semigroup is a category with topologies, in addition, Propositions 10 and 11 will show the generated topologies compatible with adjoint equivalences and bijective half-functors (which are generalized inverse operations in an n-preadditive category, see [1]). Thus, we can believe that there are natural connections between topological and algebraic structures in category structure. In this direction, we shall propose several papers and this note will be an introduction of them.

2. In order to write smoothly, we collect some facts which will be used in this note as follows:

In [1], we have obtained the following facts:

2.1. Definition. Suppose both $\mathscr{A}$ and $\mathscr{B}$ are arbitrary categories and there is a functor $F: \mathscr{A} \to \mathscr{B}$. A function $\psi: \mathscr{A} \to \mathscr{B}$ is called a bijective half-functor for F, if it satisfies the following:

(1). $\forall A \in \mathrm{ob}\mathscr{A}$: $\psi(A) = FA$.

(2). $\forall A, A' \in \mathrm{ob}\mathscr{A}$: $\psi: (A, A') \to (FA, FA')$ is bijective.

(3). $\psi(fg) = (Ff)\psi(g) = \psi(f)Fg$, whenever the composition fg makes sense.

We write $\mathrm{BH}_F(\mathscr{A}, \mathscr{B})$ for the class of all bijective half-functors for F.

In an n-preadditive category the inverse operation $a \mapsto \overline{a}$ is a bijective half-functor (see [2]).

2.2. Lemma. If $\psi \in \mathrm{BH}_F(\mathscr{A}, \mathscr{B})$, then $\psi(1_A)$ is an isomorphism for each $A \in \mathrm{ob}\mathscr{A}$.

2.3. Proposition. $\mathrm{BH}_F(\mathscr{A}, \mathscr{B}) \neq \phi$ if and only if the functor F is full and faithful.

3. This paragraph is the body of the note.

3.1. Definition. A non-empty category $\mathscr{A}$ is called a category with topologies, provided that for any A, $B \in \mathrm{ob}\,\mathscr{A}$ (A, B) is equiped with a topology

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$\tau_{(A,B)}$ and the composition $(B, C)\times(A, B)\to(A, C)$ is continuous with respect to these topologies, where the topology of $(B, C)\times(A, B)$ is the product topology.

We write C. T. for a category with topologies. Clearly, a topological semigroup with an identity is a C. T. (see [3]). Sometimes we write $(\mathscr{A}, \tau=(\tau_{(A,B)})_{A,B\in \mathrm{ob}})$ for the C. T. $\mathscr{A}$.

3.2. Lemma. Let $\mathscr{A}$ be a category, if $h: A\to A'$ and $h': B\to B'$ are isomorphisms, then when (A, B) is a topological space with a topology $\tau_{(A,B)}$ the right-left-translation T: $f\to h'fh^{-1}$ creates a topology $\{T(U)\}_{U\in\tau_{(A,B)}}$ of (A', B') and T is a homeomorphism between the two topologies.

Observing the translation T is bijective, the proof can easily be completed.

We call the above two topologies are isomorphic. Clearly, two isomorphic topologies must be homeomorphic.

Suppose $\mathscr{A}$ is a C. T., and $\tau=(\tau_{(A,B)})_{A,B\in\mathrm{ob}\mathscr{A}}$ is its topology. For any $A\in$ ob$\mathscr{A}$, pick out an object A' such that there is an isomorphism h_A: $A'\to A$, then $\tau'_{(A,B)}=\{h_B U h_A^{-1}\}_{U\in\tau_{(A',B')}}$ is a topology of (A, B), we call τ and τ' are isomorphic.

3.3. Definition. Let $\mathscr{A}$ be a C. T. and F: $\mathscr{A}\to\mathscr{B}$ a full functor. Suppose that for $B, B'\in\mathrm{ob}\mathscr{B}$, there are two objects $A, A'\in\mathrm{ob}\,\mathscr{A}$ such that $FA\cong B$, $FA'\cong B'$, and h: $FA\to B$ and h': $FA'\to B'$ are isomorphisms. The topology $\tau_{(B,B')}$ generated by the topological subbasis $\{h'FUh^{-1}\}U\in\tau_{(A,A')}$ is called a topology generated by F with respect to $\tau_{(A,A')}$. For $\psi\in\mathrm{BH}_F(\mathscr{A}, \mathscr{B})$, an analogous definition can be given.

From now on, $t_{(FA,FA')}$ denotes a topology generated by F when h' and h are identities, $\tau_{(FA,FA')}$ denotes a generated topology with any isomorphisms h and h'.

If $\{FU\mid U\in\tau_{(A,A')}\}$ is a topology of (FA, FA'), we call it a derived topology of F with respect to the topology $\tau_{(A,A')}$.

Easy to prove that if F is full and faithful then $\{FU\mid U\in\tau_{(A,A')}\}$ is a topology.

3.4. Lemma. If $\mathscr{A}$ is a C. T. and F: $\mathscr{A}\to\mathscr{B}$ is a full functor, and if there are two isomorphisms h: $FA\cong B$, h': $FA'\cong B'$, then the following are equivalent:

(1) $\tau_{(B,B')}$ is the topology generated by F with respect to $\tau_{(A,A')}$.

(2) $\tau_{(B,B')}=\bigcap\{\tau\mid\tau$ is a topology of (B, B') which contains $\{h'F(U)h^{-1}\mid U\in\tau_{(A,A')}\}\}$.

(3) $\tau_{(B,B')}=\{h'Vh^{-1}\mid V\in t_{(FA,FA')}\}$.

The crux of the proof is to show that

$$F(U_1)\cap F(U_2)=h'^{-1}(h'F(U_1)h^{-1}\cap h'F(U_2)h^{-1})h.$$

Thus, we have $U\in\tau_{(B,B')}\vdash U=\bigcup_i\bigcap_{j=1} h'F(U_{ij})h^{-1}\vdash h'^{-1}Uh=h'^{-1}(\bigcup\bigcap h'F(U_{ij})h^{-1}=\bigcup h'^{-1}(\bigcap h'F(U_{ij})h^{-1})h=\bigcup\bigcap F(U_{ij})\in t_{(FA,FA')}$, and $V\in t_{(FA,FA')}\vdash V=\bigcup\bigcap F(U_{ij})\vdash h'Vh$ $h'Vh^{-1}=h'(\bigcup\bigcap F(U_{ij})h^{-1}=\bigcup(h'\bigcap F(U_{ij})h^{-1})=\bigcup(\bigcap h'F(U_{ij})h^{-1})\in\tau_{(B,B')}$.

3.5. Definition. Functor $F: \mathscr{A}\to\mathscr{B}$ is called to be quasi-full on objects, provided that for each $B\in\text{ob}\,\mathscr{B}$ there is an object $A\in\text{ob}\,\mathscr{A}$ such that $FA\cong B$.

3.6. Definition. Given two C.T., $\mathscr{A}$ and $\mathscr{B}$, functor $F: \mathscr{A}\to\mathscr{B}$ is called a continuous mapping, if $F:(A,A')\to(FA,FA')$ is continuous with respect to the two topologies.

3.7. Proposition. If $(\mathscr{A},\tau)$ is a C.T., and topologies τ and τ' are isomorphic, then so is $(\mathscr{A},\tau')$.

3.8. Proposition. If $(\mathscr{A},\tau=(\tau_{(A,A')})_{A,A'\in\text{ob}\mathscr{A}})$ is a C.T., and if $F:\mathscr{A}\to\mathscr{B}$ is a quasi-full on objects and full functor which is a continuous mapping with respect to t, then the generated topology by F with respect to τ makes $\mathscr{B}$ a C.T..

Proof. Since F is quasi-full on objects, by the axiom of choice, for each $B\in\text{ob}\mathscr{B}$, we can pick up an object $A_B\in\text{ob}\,\mathscr{A}$ such that there is an isomorphism $h_B: FA_B\cong B$.

To start with, we are going to show that the composition $(FA_{B'},FA_{B''})\times(FA_B,FA_{B'})\to(FA_B,FA_{B''})$ is continuous, the topologies are the product topology of $t_{(FA_{B'},FA_{B''})}$ and $t_{(FA_B,FA_{B'})}$ and the topology $t_{(FA_B,FA_{B''})}$.

Since $\mathscr{A}$ is a C.T., for any $(g.f)\in(A_{B'},A_{B''})\times(A_B,A_{B'})$ and any open neighborhood U_{gf} of gf, there are open neighborhood U_g of g and open neighborhood U_f of f such that $U_g\cdot U_f\subset U_{gf}$, so that $F(U_g\cdot U_f)=F(U_g)\cdot F(U_f)\subset F(U_{gf})=U_{Fgf}$, U_{Fgf} is an open neighborhood of Fgf.

On the other hand, given an open neighborhood V_{Fgf} of Fgf, we have $F(U_{gf})\cap V_{Fgf}\neq\phi$ for any open neighborhood U_{gf} of gf, so since $F(U_{gf})\cap V_{gf}$ is open and F is a continuous mapping, $F^{-1}(F(U_{gf})\cap V_{Fgf})=V_{gf}$ is an open neighborhood of gf, in addition, $F(V_{gf})\subset F(U_{gf})\cap V_{Fgf}\subset V_{Fgf}$, therefore, according to the above statement, there are two open neighborhoods U_g and U_f such that $FU_g\cdot FU_f\subset F(V_{gf})\subset V_{Fgf}$.

Secondly, we are going to prove the composition $(B',B'')\times(B,B')\to(B,B'')$ is continuous.

By the axiom of choice, we can pick up isomorphisms h: $FA_B\cong B$, h': $FA_{B'}\cong B'$ and h'': $FAB''\cong B''$. Given $(f',g')\in(B',B'')\times(B,B')$ and an open neighborhood $U_{f'g'}$ of $f'g'$, we know there is an open set $V\in t_{(FA_B,FA_{B''})}$ such that $h''Vh^{-1} = U_{f'g'}$ by Lemma 3.4. Hence there is a unique $v \in V$ such that $f'g' = h''vh^{-1}$, so that $(h''^{-1}f'h')(h'^{-1}g'h) = v$. Let $f_o = h''^{-1}f'h'$ and $g_o = h'^{-1}g'h$, we have $f_o \in (FA_{B'}, FA_{B''})$, $g_o \in (FA_B, FA_{B'})$ and $f_o g_o = v \in V$. By the first part of the proof, there are open neighborhood V_{f_0} of f_o and open neighborhood V_{g_0} of g_o such that $_{f_0} \cdot V_{g_0} \subset V$. Hence $(h''V_{f_0}h'^{-1}) \cdot (h'V_{g_0}h^{-1}) \subset h''Vh^{-1} = U_{f'g'}$. Lemma 3.4 shows $h''V_{f_0}h'^{-1}$ and $h'V_{g_0}h^{-1}$ are open in $\tau_{(B', B'')}$ and $\tau_{(B, B')}$ respectively, so since $f' \in h''V_{f_0}h'^{-1}$ and $g' \in h'V_{g_0}h^{-1}$, the proof is complete.

3.9. Proposition. Let $\mathscr{A}$ be a C. T. and $F: \mathscr{A} \to \mathscr{B}$ a full functor, then $F: (A_1, A_2) \to ((FA_1, FA_2), t_{(FA_1, FA_2)})$ is homeomorphic if and only if F is full and faithful.

3.10. Proposition. If $\mathscr{A}$ is a C. T., and if $\psi \in \mathrm{BH}_F(\mathscr{A}, \mathscr{B})$, then the topology generated dy ψ is isomorphic to the topology generated by F.

Referring to the paragraph 3.3, the proof can easily be completed by Lemma 2.2, Proposition 2.3 and Lemma 3.2.

3.11. Proposition. If $(\mathscr{A}, \tau)$ is a C. T. and $\langle T, S; \eta, \in \rangle: \mathscr{A} \to \mathscr{B}$ is an adjoint equivalence, then the topology generated by S with respect to the topology generated by T is isomorphic to the topology τ of $\mathscr{A}$.

[4, 3. Th.1 and 4.] show T and S are full and faithful, then the paragraph 3.3 and Lemma 3.2 complete the proof.

References

[1] Yu Yong Xi, On Bijective Half-functors, to appear.

[2] Yu Yong Xi, The Quasikernels in an n-preadditive Category, J. Math. Research and Exposition, first issue (1981), 7—15.

[3] J. H. Carruth, J. A. Hildebrant, R. J. Koch, The Theory of Topological Semigroups, New York and Basel (1983).

[4] S. Mac Lane, Categories for the Working Mathematician, New York-Heidelberg-Berlin, Springer (1971).