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Solvable CN-Groups and π-Separable $C\pi\pi$-Groups*

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G. Higman studied first the finite groups in which every element has prime power order except 1 (see[1]), that is, the centralizer of every element is is a p-group except 1. Later many authors have generalized it. On the one hand, the generalization is CN-groups, that is, the finite groups in which the centralizer of every element is nilpotent except 1 (see[2, 3]). On the other hand, the generalization is C22-groups, that is, the groups are of even order and the centralizer of any involution is a 2-group; a C22-group named again CIT-group (see[3]). [3] showed that a nonsolvable CN-group and a nonslvable CIT group are identical and classified such groups completely. About solvable CN-groups[2, 13] have had a discussion. In recent years Z. Arad and other authors generlized a C22-group to a Cpp-group G, i.e. the centralizer of any non-identity p-element is a p-group for some prime $p||G|$. They further generlized to a $C\pi\pi$-group, i.e. the centralizer of any non-identity π-element is a π-group of G, in which π is a nonempty proper subset of all prime divisors of $|G|$. About Cpp-groups[5,6] classified such groups completely for $p=3$. About $C\pi\pi$-groups[14, 15, 16] have had a discussion when it is π-solvable.

The main content of this paper is to continue a discussion of solvable CN-groups and π-separable $C\pi\pi$-groups. The results indicate that they possess roughly the same structure. This paper apply mainly the theory of fixed-point-free actions, thus the obtained results are more extensive and detailed than predecessors. In addition, we have produced the sufficient and necessary condition which CN-groups and $C\pi\pi$-groups are either solvable or supersolvable.

I. Solvable CN-Groups

Lemma 1. Let G be a CN-group, H_1 and H_2 be nilpotent subgroups of G, $(|H_1|, |H_2|)=1$. If there are $1\neq x_i\in H_i$, $i=1,2$, satisfy $x_1x_2=x_2x_1$, then H_1 and H_2 permute elementwise.

This lemma is a generalization of [2] Lemma 2.3. Their proofs are all the same. By Lemma 1, the maximal nilpotent subgroups of G are Hall subgroups.

Lemma 2. Let G be a CN-group, if the 2-sylow subgroup of G is a generalized

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quaternion group, then G has a normal 2-complement, thereby G is solvable.

Proof. Since the 2-sylow subgroups of G are generalized quaternion groups, by[7] G possesses a maximal odd normal subgroup G_1. Thus it causes G/G_1 to have a central element a of order 2. But since G_1 is an odd order group, it is solvable. By [2] Lemma 1.5 we see that G/G_1 is a CN-group, thus $G/G_1 = C_{G/G_1}(a)$ is aanilpotent group. Furthermore, from the maximum property of G_1 we see that G/G_1 is a 2-group, hence G has a normal 2-complement G_1, therefore G is solvable.

As pointed out in the proof of [2] Lemma 1.8, Burnside's theorem on fixed-point-free actions ([8], p336 TheoremV) is false in general, but it is true for CN-groups.

Theorem 1. Let CN-group H act fixed-point-free on a group $K \neq 1$, then H is either a cyclic group or the direct product of a generalized quaternion group and a cyclic group of odd order.

Proof. By[9] Theorem 7.24, every sylowsubgroup of H is a cyclic group or a generalized quaternion group. When the 2-sylow subgroup S_2 of H is cyclic, then by Burnside's theorem ([10] Theorem 14.3.1), H has a normal 2-complement, therefore H is solvable. When S_2 is a generalized quaternion group, also by Lemma 2 H is solvable. Thus H has a $p_1 p_2$-Hall subgroup L for any $p_1 p_2 \mid |H|$. Let $p_1 < p_2$. If p_1, p_2-sylow subgroups are cyclic, then $P_2 \lhd P_1P_2 = L$. Hence if a is an element of order p_1 of P_1 and b is an element of order p_2 of P_2, then L has the subgroup $\langle a, b\rangle$ of order $p_1 p_2$, moreover $\langle b\rangle \lhd \langle a, b\rangle$. If a and b do not permute, since $\langle b\rangle \cap K = 1$, then a does not permute with the elements of K except 1 either. Thus $\langle a\rangle$ acts fixed-point-free on $\langle b\rangle K$, by Thompson's theorem ([9]Theorem 12.9), $\langle b\rangle K$ is nilpotent, contrary to that $\langle b\rangle$ acts fixed-point-free on K. Therefore a permutes with b, and by Lemma 1, $L = P_1 \times P_2$. If P_1 is a generalized quaternion group, then by Lemma 2 $P_2 \lhd P_1P_2 = L$. Let a be an element of order 2 of P_1, b be an element of order p_2 of P_2. Since P_2 is a cyclic group, its subgroups are characteristic subgroups. We see that $\langle b\rangle \lhd L$ and we obtain the group $\langle a\rangle\langle b\rangle$ of order $2p$. As the discussion above we conclude that a and b permute, $L = P_1 \times P_2$. Hence the elements of Sylow subgroups of coprime orders of H permute. Therefore H is nilpotent, and the theorem is proved.

Theorem 2. The Fitting subgroup F of CN-group G is either a Hall subgroup of G or a p-group. If G is nonsolvable, then $F = 1$ or F is a 2-group.

I. When $F \neq 1$ is a Hall subgroup of G, then $G = F$ or $G = HF$, $H \cap F = 1$, H is a cyclic group or the direct product of a generalized quaternion group and a cyclic group of odd order. Furthermore, G is a Frobenius group with Frobenius kernel F.

Ⅱ. When G is solvable and F is not a Hall subgroup of G, then F is a p-group, G = PHF, HF$\lhd$G, in which P is a cyclic p-group; H is a cyclic p'-group. Furthermore, PH is a Frobenius group with Frobenius kernel H, HF is a Frobenius group with Frobenius kernel F. Also if $p_1, p_2, \cdots, p_r$ a all different prime factors of $|H|$, then $|P|\,|\,(p_1-1, p_2-1, \cdots, p_r-1)$.

Proof. If F is not a p-group, let $q_1\,|\,|F|$, then $|F|$ possesses another prime factor q_2 at least. We write Q_1, Q_2 as q_1, q_2-sylow subgroups of G separately, $O_{q_i}(G)$ for maximal normal q_i-subgroup of G, $D = Z(O_{q_2}(G))$. Since $F = O_{q_1}(G) \times O_{q_2}(G) \times \cdots$, by Lemma 1, thus any element of Q_1 permute any element of D, that is, $C_G(D) > Q_1$. Moreover, since D is a characteristic subgroup of $O_{q_2}(G)$, thus it is a characteristic subgroup F, we conclude that $D \lhd G$, hence $C_G(D) \lhd N_G(D) = G$. But since G is a CN^*-group and $C_G(D)$ is a nilpotent group, $D < Z(F)$, we see that $C_G(D) = F$. Thus $F > Q_1$, therefore F is a Hall subgroup of G.

when $F \neq 1$, if F is a Hall subgroup of G, and $F \neq G$, since $F \lhd G$, then G splits over F, *i.e.* $G = HF$, $H \cap F = 1$, and since $C_G(F)$ is a nilpotent normal subgroup of G, $C_G(F) \leq F$, therefore F is a maximal nilpotent subgroup of G. Hence the elements of H does not permute with the elements of F except 1, *i.e.* $C_G(y) \leq F$, $1 \neq y \in F$, therefore H acts fixed-point-free on F. From theorem 1, H is either a cyclic group or the direct product of a generalized quaternion group and a cyclic group of odd order, again by [9] Theorem 10.5, G is a Frobenius group with Frobenius kernel F. In this case G is obviously solvable. Suppose F is a p-group, as the proof above we show easily that any p-element and p'-element do not permute. If $p \neq 2$, then as G is a group of even order and 2-sylow subgroup S_2 of G is a generalized quaternion group, by Lemma mma 2, G is solvable. Suppose S_2 is not a generalized quaternion group, then since S_2 acts fixed-point-free on F, we infer that S_2 is a cyclic group. Moreover, by Burnside's theorem ([10] Theorem 14.3.1), G has a normal 2-complement, thus G is solvable. Therefore, if G is nonsolvable, then F = 1 or F is a 2-group.

When G is solvable and F is not a Hall subgroup of G, we may assume that F is a p-group. Let us write H as p-complement of G, H acts fixed-point-free on F, by Theorem 1, H is a cyclic group or the direct product of a generalized quaternion group and a cyclic group of odd order. Since F is a maximal p-subgroup, the minimal normal subgroup $\overline{P_i}$ of G/F is not a p-group, we may assume that HF/F contains $\overline{P_i}$ ([10] Theorem 9.3.1), also since H is nilpotent group, then $HF/F \leq C_{G/F}(\overline{P_i})$. But since G/F is a CN^*-group, H is a maximal nilpotent subgroup of G, we conclude that HF/F is a maximal nil-nilpotent subgroup of G/F, therefore.

$$C_{G/F}(\overline{P}_i) = HF/F \lhd N_{G/F}(\overline{P}_i) = G/F$$

Thus $HF \lhd G$. Since H is a cyclic group or the direct product of a generalized quaternion group and a cyclic group of odd order, also since $\overline{P}_i$ is contained in HF/F, and $HF/F \cong H$, hence $\overline{P}_i$ is a group of prime order. $|\overline{P}_i| = p_i \mid |H|$, but

$$G/HF \cong \frac{G/F}{HF/F} = \frac{N_{G/F}(\overline{P}_i)}{C_{G/F}(\overline{P}_i)} \cong \text{a subgroup of a cyclic group of order } p_i - 1,$$

we conclude that G/HF is a cyclic p-group. Furthermore, $|G/HF| \mid (p_1 - 1, p_2 - 1, \cdots, p_r - 1)$, in which $p_1, p_2, \cdots, p_r$ are different prime factors of $|HF/F| = |H|$. Since G/HF is a cyclic p-group, from $p \mid (p_1 - 1, \cdots, p_r - 1)$ we see that p is the minimal prime factor of $|G|$, thus H is a group of odd order, it must be a cyclic p'-group. As the reason above, HF is a Frobenius group with Frobenius kernel F. Since HF is a solvable normal subgroup of G, H is a Hall subgroup of HF, by Frattini's argument ([11]Proposition IV. 2.e), $G = N_G(H)HF = N_G(H)F$. Also from $N_G(H) \cap F \lhd N_G(H)$, we obtain that the elements of $N_G(H) \cap F$ permute with the elements of H, hence $N_G(H) \cap F = 1$. Let P is a p-sylow subgroup of $N_G(H)$, thus $N_G(H) = PH$, $G = PHF$. $P \cong PH/H \cong \frac{PHF/F}{HF/F} \cong PHF/HF = G/HF$, P is a cyclic group. Since the p-elements and p'-elements of G do not permute, hence PH is a Frobenius group with Frobenius kernel H.

This theorem is a generalization of [1]Theorem 1, [2]Lemma1.8, [13] p402, Theorem 1.5.

Corollary 1. If G is a 3-step group, then $G = PHF$, in which the meanings of P, H, F is stated in Theorem 2. Ⅱ.

Proof. From[13] p401, Lemma 1.4, any 3-step group is a solvable CN-group, Again by Theorem 2, this completes the proof.

Corollary 2. CN-group G is solvable if, and only if, G has a 2-complement and the Fitting subgroup $F \neq 1$ of G.

Proof. It suffices to show sufficiency. Moreover, by Theorem 2. we need only show when F is a 2-group. Since $C_G(F)$ is a nilpotent normal subgroup of G, $C_G(F) \leq F$, hence a 2-complement H in G acts fixed-point-free on F. By Theorem 1. H is a cyclic group and $G = HS_2$, in which S_2 is a 2-sylow subgroup of G. By Wielandt-Kegel's theorem ([11] IX. 2.e), G is a solvable group.

To further understand the construction of the solvable CN-groups, first ws give the following lemma.

Lemma 3. Let group $G = PQ$, in which P is a cyclic group, Q is a minimal normal subgroup of G, and Q is a sylow subgroup of G. If $C_G(Q)=Q$, $|Q|=q^\beta$, then β is the exponent of $q \pmod{|P|}$.

Proof. Since Q is an elementary Abelian group of order q^β, and $G=PQ$, from Q is a minimal normal subgroup of G we see that Q is also a minimal P invariant subgroup of G. Whence $\rho: P \to GL(Q)$ is an irreducible representation (GL(Q) is an automorphism group of Q, it is a full linear group). By $C_G(Q)=Q$ we see that the representation is faithful. Let $P=\langle a\rangle$, then the vector space Q does not contain a proper $\rho(a)$ invariant subspace, Otherwise it is contrary to that Q is an irreducible P module. Let the matrix corresponding $\rho(a)$ be A on a certain basis of Q, then by[12] Chapter 3, Therem 2 we see that the characteristic polynmial of A is a minimal polynomial. Also if the order of A is m, then $A^m=I$, that is, the characteristic polynomial of A divides exactly x^m-1. Hence the characteristic roots of A are all unit roots of degree m on a certain finite extension field $K(q^n)$ of a q element field C_q, moreover, there is not a repeated root. Thus if $f(x) \pmod q$ could be reduced, $f(x)=f_1(x)f_2(x)$, we would prove easily that

$$A \sim \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$$

on C_q, in which A_1, A_2 are respectively take $f_1(x), f_2(x)$ as a characteristic polynomial. Since A is irreducible, therefore $f(x) \pmod q$ is also irreducible.

Let $\omega \in K(q^n)$, $f(\omega)=0$, thus the number of the conjugate of ω is identical with the degree β of $f(x)$. Suppose ξ is a generator of $K(q^n)$, thus $\xi \mapsto \xi^{q}$ generates the automorphism group of $K(q^n)$, Under this automorphism, $\omega \mapsto \omega^q$. As the order of A is m, the order of ω is also m, $m=|P|$. Hence the all conjugates of ω are $\omega^{q^0}, \omega^{q^1}, \cdots, \omega^{q^{\gamma-1}}$, in which γ is the exponent of $q \pmod{|P|}$, therefore $\beta=\gamma$ and the lemma is proved.

Theorem 3. Let G be a solvable CN-group, and let G be not nilpotent; let F be the Fitting subgroup of G.

I. If F is a Hall subgroup of G, then $G=HF$, $H\cap F=1$. Let $H=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $F=q_1^{\beta_1}\cdots q_s^{\beta_s}$.

1. When H is a cyclic group, then G possesses the chief factors

$$\underbrace{p_1, \cdots, p_1}_{a_1}, \cdots, \underbrace{p_r, \cdots, p_r}_{a_r};\ \underbrace{q_1^{b_1}, \cdots, q_1^{b_1}}_{\omega_1}, \cdots, \underbrace{q_s^{b_s}, \cdots, q_s^{b_s}}_{\omega_s}.$$

in which b_i is the exponent of $q_i \pmod{|H|}$, $\omega_i b_i=\beta_i\ (i=1,\cdots,s)$. Moreover the class of F is not more that $\sum_{i=1}^{s}\omega_i$.

2. When H is the direct product of a generalized quaternion group and

a cyclic group of odd order, then G possesses the chief factors

$$\underbrace{2,\cdots,2}_{a_1};\ \underbrace{p_2,\cdots,p_2}_{a_2},\cdots,\underbrace{p_r,\cdots,p_r}_{a_r};\ q_1^{b_{11}},\cdots,q_1^{b_{1\omega_1}},\cdots,q_s^{b_{s1}},\cdots,q_s^{b_{s\omega_s}}.$$

in which $b_i | b_{ij}$, b_i is the exponent of $q_i(\bmod |H_{2'}|)$, $\sum_{j=1}^{\omega_i} b_{ij}=\beta_i$, $b_{ij} \geq 1$ $(i=1,\cdots,s, j=1,\cdots,\omega_i)$. Moreover, the class of F is not more than $\sum_{i=1}^{s}\omega_i$.

Ⅱ. If F is not a Hall subgroup of G, then G = PHF. Let $|P|=q^{\gamma}$, $|H|=p_1^{a_1}\cdots p_r^{a_r}$, $|F|=q^{\beta-\gamma}$, $0<\gamma<\beta$, we have $q^{\gamma} | (p_1-1,\cdots,p_r-1)$. Moreover, G possesses the chief factors

$$\underbrace{q,\cdots,q}_{\gamma};\ \underbrace{p_1,\cdots,p_1}_{a_1},\cdots,\underbrace{p_r,\cdots,p_r}_{a_r};\ q^{d_1},\cdots q^{d_k}.$$

in which $\sum_{i=1}^{k} d_i=\beta-\gamma$, $\gamma<d$, $d | d_i$, $i=1,\cdots,k$. and d is the exponent of $q\ (\bmod |H|)$.

Proof. Ⅰ. If F is a Hall subgroup of G, then by Theorem 2, G = HF, H is a cyclic group or the direct product of a generalized quaternion group and a cyclic groupp of odd order. Since F is a nilpotent group, the Hall subgroups of F are normal in G, hence G has a chief series

$$G>\cdots>F>\cdots>C_t>1,$$

in which C_t is an elementary Abelian group of order $q_i^{b_i}$, and q_i can be any number among $q_1,\cdots,q_s$. We conside the subgroup $G_t=HC_t$, where C_t is a minimal normal s subgroup of G_t. If not, we assume $1\neq C_n<C_t$, $C_n \lhd C_t$, then by $Z(F) \lhd G$, $1\neq Z(F)\cap C_t \lhd G$ we see that $C_t \leq Z(F)$, thereby $C_n \lhd F$. Hence $C_n \lhd \langle HC_t, F\rangle = HF = G$, contrary to that C_t is a minimal normal subgroup of G.

When H is a cyclic group, by Lemma 3, b_i is t the exponent of $q_i(\bmod |H|)$. Considering the factor groups, by induction we see that G possesses the chief factors

$$\underbrace{p_1,\cdots,p_1}_{a_1},\cdots,\underbrace{p_r,\cdots,p_r}_{a_r};\ \underbrace{q_1^{b_1},\cdots,q_1^{b_1}}_{\omega_1},\cdots,\underbrace{q_s^{b_s},\cdots,q_s^{b_s}}_{\omega_s},$$

in which $\omega_i b_i=\beta_i (i=1,\cdots,s)$ and the class of F is not more than $\sum_{i=1}^{s}\omega_i$

When H is the direct product of a generalized quaternion group and a cyclic group of odd order, then by Lemma 2, G has a normal 2-complement G_1. Suppose P is a cyclic group of of order 2^{a_1-1} of the 2-sylow subgroup S_2 of G $(|S_2|=2^{a_1})$, $G_2=PG_1$, thus G has a normal series

$$G>G_2>G_1\geq F>1.$$

Applying above discussion to G_2, we conclude that G possesses the chief factors

$$\underbrace{2,\cdots,2}_{\alpha_1};\ \underbrace{p_2,\cdots,p_2}_{\alpha_2},\cdots,\underbrace{p_r,\cdots,p_r}_{\alpha_r};\ q_1^{b_{11}},\cdots,q_1^{b_{1\omega_1}},\cdots,q_s^{b_{s1}},\cdots,q_s^{b_{s\omega_s}}.$$

in which $b_i | b_{ij}$ ($j=1,\cdots,\omega_i$, $i=1,\cdots,s$), b_i is the exponent of $q_i (\mathrm{mod}\ |H/2|)$, $\sum_{j=1}^{\omega_i} b_{ij}=\beta_i$ ($i=1,\cdots,s$), moreover, the class of F is not more than $\sum_{i=1}^{s}\omega_i$.

Now we prove $b_{ij}>1$, it must only show $b_{s\omega_s}>1$. Let C_t is a minimal normal subgroup of G, if $|C_t|=q$, we discuss S_2C_t, by Lemma 1, the centralizer of C_t in S_2C_t is itself. Also since the automorphism group of C_t is a cyclic group and S_2 is a generalized quaternion group, we see that this case can not happen.

Ⅱ. If F is not a Hall subgroup of G, by Theorem 2, we can assume that F is a q-group, Moreover by $|P|=q^\gamma$, we obtain that $q^\gamma | (p_1-1, \cdots, p_r-1)$. In this case G has a normal series

$$G=PHF>HF>F>1,$$

in which P and H are the cyclic groups of coprime orders. we refine it so as to obtain a chief series

$$G>\cdots>HF>\cdots>F>\cdots>C_t>1.$$

Its chief factors are $\underbrace{q,\cdots,q}_{\gamma};\ \underbrace{p_1,\cdots,p_1}_{\alpha_1},\cdots,\underbrace{p_r,\cdots,p_r}_{\alpha_r};\ q^{d_1},\cdots,q^{d_k}$, in which $\sum_{i=1}^{k} d_i=\beta-\gamma$. Discussing the normal series of HF, since F is a normal Hall subgroup of HF, therefore the chief factors of HF are $\underbrace{p_1,\cdots,p_1}_{\alpha_1},\cdots,$ $\underbrace{p_r,\cdots,p_r}_{\alpha_r};q^d,\cdots,q^d$, in which d is the exponent of $q(\mathrm{mod}\ |H|)$. Hence we refine the normal series $HF>\cdots>F>\cdots>C_t>1$ of HF so as to obtain a chief series, we obtain that $d|d_i$, $i=1,\cdots,k$. Since $q^\gamma | (p_1-1,\cdots,p_r-1)$, also since the elements of L_h in semi-direct product L_hC_m (L_h is a subgroup of orden p_h of H; C_m is a minimal normal subgroup of HF) do not permute with the elements of C_m except 1, hence the normalizer of L_h in L_hC_m is itself, thus $1+kp_k=|C_m|=q^d$, we conclude that $p_h|q^d-1$, $h=1,\cdots,r$. Therefore $q^\gamma\le p_h-1<p_h\le q^d-1<q^d$, hence $\gamma<d$.

Theorem 4. A CN-group G is supersolvable if, and only if, $F\neq 1$ and there is a normal subgroup of order q in G, for all $q||F|$, in which F is a Fitting subgroup of G.

Proof. It suffices to show sufficiency. If $|F|$ is an even number, then by the hypothesis G has a normal subgroup of order 2, G possesses a central element of order 2, from G is a CN-group we see that G is a nilpotent group. Clearly G is supersolvable.

If $|F|$ is an odd number, then by Theorem 2, G is solvable. Moreover since G has a normal subgroup Q of order q, for all $q \mid |F|$, it lies in the center of the q-sylow subgroup Q_1 of G. Thus $C_G(Q) \geq Q_1$, but $C_G(Q)$ is a nilpotent normal subgroup of G, hence $F \geq C_G(Q) \geq Q_1$. Therefore F is a Hall subgroup of G. Since q is all a chief factor of G, for all $q \mid |F|$, by the uniqueness of the chief factors and Theorem 3. I., we conclude that G possesses the chief factors $\underbrace{p_1, \cdots, p_1}_{\alpha_1}, \cdots, \underbrace{p_r, \cdots, p_r}_{\alpha_r};$ $\underbrace{q_1, \cdots, q_1}_{\beta_1}, \cdots \underbrace{q_s, \cdots q_s}_{\beta_s}$. G is a supersolvable group.

2. π-Separable Cππ-Groups

From the definition of Cππ groups we see that G is a Cππ-group if, and only if, G does not contain (π, π')-mixed elements, that is, the non identity element in G is either a π-element or a π′-element. Thus, that G is a Cππ-group but G is not a π-group is equal to that G is a Cπ′π′-group but G is not a π′-group.

Lemma 4. Let G be a Cππ group, $A \lhd G$, if G/A has a non-identity π-element, then G/A is a C ππ-group.

Proof. Since G is a Cππ group, G has no (π, π')-mixed elements, this property is kept under the homomorphism. In fact, if G/A is not a Cππ group, since G/A has a non-identity π-element, G/A is not a π′-group, Then G/A has a (π, π')-mixed element $\bar{g}$. Suppose g is an inverse image of $\bar{g}$ in G, since under the homomorphism $|\bar{g}| \mid |g|$, we conclude that g is a (π, π')-mixed element, contrary to that G is a Cππ-group.

Lemma 5. If G is a Cππ-group, G is not a π group, $1 \neq A \lhd G$ and then $C_G(A)$ is a nilpotent normal subgroup of G.

Proof. Since $C_G(A) \lhd N_G(A) = G$, if $C_G(A) = 1$, then the conclusion is obviously true. Thus we can assume $C_G(A) \neq 1$, by $C_G(A)$ is a Cππ-group, G can only contain π-elements or π′-elements, thereby $C_G(A)$ is either a π-grouq or a π′-group. If $C_G(A)$ is a π-group, then since G is not a π-group, $C_G(A)$ admits a fixed-point-free automorphism of order q ($q \in \pi'$), by Thompson's theorem ([9] Theorem 12.9), $C_G(A)$ is a nilpotent group. If $C_G(A)$ is a π′-group, then since G is a Cππ-group, $C_G(A)$ admits a fixed-point-free automorphism of order p ($p \in \pi$), it is similarly nilpotent.

For convenience, we call the finite groups in which every Sylow subgroup is a cyclic group or a generalized quaternion group ZQ-groups. Obvionsly, the subgroups of ZQ-groups are ZQ-groups.

Lemma 6. If ZQ-group N is a characteristic simple group, then the order of N is a prime.

Proof. The characteristic simple group is the direct product of isomorphic simple groups, its every direct factor is a simple ZQ-group, by [7], a simple ZQ-group is a group of prime order. Again because every Sylow subgroup of ZQ-groups contains only a group of order p, we conclude that the order of N is a prime.

Theorem 5. Let G be a π separable group and let G be not a π-group; let F be a fitting subgroup of G. Thus

I. F is a non-identity π'-group.

1. When F is a π'-Hall subgroup of G, $G=HF$, H is a ZQ-group. Moreover, G is a Frobenius group with Frobenius kernel F.

2. When F is not a π'-Hall subgroup of G, $G=KHF$, in whick K is a cyclic π'-group, H is a cyclic π-group of odd order, $F=O_{\pi}(G)$. $|K|\,|\,(p_1-1,\cdots,p_r-1)$, $p_1,\cdots,p_r$ are all different prime factors of $|H|$. KH is a complementary group of F, moreover, it is a Frobenius group with Frobenius kernel H. $HF\lhd G$, HF is a Frobenius group with Frobenius kernel F. or

II. F is a non-identity π-group. In this case we can suppose that H is a π'-Hall subgroup of G. and we have the same conclusion to I (exchanging only π and π').

Proof. Since G is a π-separable $C\pi\pi$-group, and G is not a π-group, hence, $O_{\pi}(G)$ or $O_{\pi'}(G)$ is not 1 and G. But since G is a π'-separable $C\pi'\pi'$-group and G is not a π'-group, therefore II and I have completely the same conclusion. we need only discuss the case of $O_{\pi'}(G)\neq 1$.

Since G is a $C\pi\pi$-group, $O_{\pi'}(G)$ admits a fixed-point-free automorphism of order p ($p\in\pi$), by Thompson's theorem ([9] Theorem 12.9) $O_{\pi'}(G)$ is a nilpotent group, hence $F=O_{\pi'}(G)$. If F is a π'-Hall subgroup, then G has π' complement H, H is a π-Hall subgroup. It acts fixed-point-free on π'-group F. Hence H is a ZQ-group and case I. 1, is true. Let F be not a π'-Hall subgroup, by the maximum property of F, $\overline{G}=G/F$ does not contain the normal π-subgroup, but $\overline{G}$ is π-separable, it must contain a minimal normal π-subgroup $\overline{P}$, $\overline{P}=PF/F\cong P$. Since π-group P acts fixed-point-free on π'-group F, $\overline{P}$ is a ZQ-group. And by Lemma 6, $\overline{P}$ is a group of order p ($p\in\pi$). Moreover, since $\overline{G}$ is a $C\pi\pi$-group, hence $C_{\overline{G}}(\overline{P})$ does not contain π'-elements, it is a π-group. Furthermore, from F is not a π'-Hall subgroup of G, $\overline{G}$ is not a π-group and

$N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})=\overline{G}/C_{\overline{G}}(\overline{P})\lesssim$ (the subgroup of $\mathrm{Aut}(\overline{P})$, i.e. the subgroup of cyclic group of order $p-1$), we see that $\overline{G}/C_{\overline{G}}(\overline{P})$ can only contain π'-elements and can't contain π-elements, therefore $\overline{G}/C_{\overline{G}}(\overline{P})$ is a cyclic π'-group. That is, $\overline{G}/C_{\overline{G}}(\overline{P})\cong\overline{L}$, in which $\overline{L}$ is a π'-Hall subgroup of $\overline{G}$, moreover $|\overline{L}|\,|p-1$. Also by Lemma 5, $C_{\overline{G}}(\overline{P})$ is a nilpotent normal subgroup of $\overline{G}$, hence $C_{\overline{G}}(\overline{G})=F(\overline{G})$, and $\overline{G}=F(\overline{G})\overline{L}$. Since the inverse image of $F(\overline{G})$ in G has the normal Hall subgroup F, hence its complement H exist. Therefore, $F(\overline{G})=HF/F\cong H$, H is a nilpotent π-group of ZQ type and $HP\lhd G$. By Frattini's argument ([11] IV.2.e), we obtain $G=N_G(H)HF=N_G(H)F$, also since the non-identity elements between F and H do not permute, we conclude $N_G(H)\cap F=1$. Let K be a π-complement of $N_G(H)$, thus $N_G(H)=KH$, $G=KHF$, and $K\cong\overline{L}$, K is a cyclic π'-group. Suppose $p_1,\cdots,p_r$ are all different prime factors of $|H|$, since $H\cong F(\overline{G})\lhd\overline{G}$, H is a nilpotent ZQ-group, we obtain that $\overline{G}$ has the normal subgroup of order p_i $(i=1,\cdots,r)$. Like the discussion of $\overline{P}$ we can obtain $|K|\,|(p_1-1,\cdots,p_r-1)$. Hence if H is a group of even order, then we have clearly that $K=1$, $G=HF$, F is a π'-Hall subgroup of G, contrary to hypothesis. Therefore, H is a cyclic π-group of odd order.

In this theorem the conclusions about Frobenius groups can inmediately be obtained by the definition of $C\pi\pi$ groups and [9] Theorem 10.5.

This theorem is a generalization of [1] Theorem 1, [3] II. Theorem 1, Theorem 2, the partial results of [6] and [15] Lemma 2.3.

From Theorem 5 we deduce easily the conclusion of [14] that if G is a solvable $C\pi\pi$-group; G is not a π group, then either a π-Hall subgroup or a π'-Hall subgroup of G is a nilpotent CCT-subgroup. Furthermore, we can generalize following corollary from this theorem.

Corollary 3. If G is a π-separable $C\pi\pi$-group and G is not a π-group, then either a π-Hall subgroup or a π'-Hall subgroup of G is a nilpotent CCT-subgroup.

Proof. By Theorem 5, we need only conside this case in which F is a non-identity π'-group. When F is a π'-Hall subgroup of G, $G=HF$, where F is a Frobeniue kernel of G, by [9] Theorem 10.5.2) we see that F is a nilpotent CCT-subgroup. When F is not a π'-Hall subgroup of G, $G=KHF$, where the cyclic group H is a nilpotent π-Hall subgroup of G; moreover, H is a Frobenius kernel of KH and H is a Frobenius complement of HF. Let g be an element of G. Since $g=khf$, $k\in K$, $h\in H$, $f\in F$, $H^g=H^f$, hence $H\cap H^g=H\cap H^f=1$ or H, that is, H is a TI-set. Moreover, since if the element $g\neq 1$ of G permute with the element $h_1\neq 1$ of H, then we can conclude $g = h \in H$. Thus $C_G(h_1) \leq H$, $1 \neq h_1 \in H$, i.e. H is a CC subgroup. Therefore, in this case the π-Hall subgroup H is a nilpoten CCT-subgroup.

According to Theorem 5 and Lemma 3, we may discuss the chief factors of the solvable $C\pi\pi$ groups as we discuss the solvable CN-groups.

Theorem 6. Let G be a $C\pi\pi$-group and let G be not a π-group; let F be a Fitting subgroup of G. If G has a normal subgroup Q of order q, then G is solvable, and $F = C_G(Q)$, $|G/F| \mid q-1$; $G = HF$, in which H is a cyclic Hall subgroup of G. If G has a normal subgroup of order q_i, for all $q_i \mid |F|$, then G is supersolvable.

Proof. Since $Q \lhd G$, $Q \lhd O_q(G)$, we obtain that $Q \leq Z(O_q(G))$, hence $Q \leq Z(F)$, thus $C_G(Q) \geq F$. Also by Lemma 5, $C_G(Q)$ is a nilpotent normal subgroup of G, from the maximum property of F, $C_G(Q) = F$. But since

$G/F = N_G(Q)/C_G(Q)$ = the subgroup of cyclic group of order $q-1$, we obtain that G/F is solvable, deduce that G is solvable, and $|G/F| \mid q-1$. Since G is a $C\pi\pi$-group and G is not a π group, hence from G/F is an abelian group, we result in $(|G/F|, |F|) = 1$. Thereby $G = HF$. H is a cyclic Hall subgroup of G.

If G has a normal subgroup Q_i of order q_i, for all $q_i \mid |F|$, then from the discussion above we see that G is solvable. moreover $C_G(Q_i) = F$. Furthermore, $G = HF$, F is a Hall subgroup of G, and the complementary group H of F is cyclic. Like the case of Theorem 3.1.1 completely, by Lemma 3 we can immediately conclude that tht chief factors of G are $\underbrace{p_1, \cdots, p_1}_{\alpha_1}, \cdots, \underbrace{p_r, \cdots, p_r}_{\alpha_r}, \underbrace{q_1^{b_1}, \cdots q_1^{b_1}}_{\beta_1}, \cdots, \underbrace{q_s^{b_s}, \cdots q_s^{b_s}}_{\beta_s}$; in which $|H| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $|F| = q_1^{\beta_1} \cdots q_s^{\beta_s}$, b_i is the exponent of $q_i (\mathrm{mod}\ |H|)$. Since q_i is all a chief factor of G. for all $q_i \mid |F|$, by the uniqueness of the chief factors we see that G is supersolvable.

We see easily that the supersolvable conditions is this theorem are also necessary.

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