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Ramsey Numbers for Triangles and Graphs of Order Four with No Isolated Vertex

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Abstract Let mG denote the union of m vertex-disjoint copies of G.The Ramsey numbers r(mG, nK_3) are determined for all graphs of order four with no isolated vertex.

Ⅰ. Introduction

The determination of Ramsey numbers r(G, H) has been received attention from several mathematicians in recent years. There has been notable success in evaluation of exact Ramsey numbers for multiple copies of graphs (see 〔3〕、〔4〕、〔5〕). A general and extensive survey is given in 〔1〕.

Let mG denote the union of m vertex-disjoint copies of G. The Ramsey number $r(mG, nK_3)$ is then the least integer p such that if the edges of K_p are two-colored, say red and green, there must be either a "red mG"(a graph mG with all edges colored red) or a "green nK_3"; its existence is guaranteed by famous theorem of Ramsey. It is known that the Ramsey numbers $r(mK_3, nK_3)$ and $r(mK_4, nK_3)$ are given by S. A. Burr, P. Erdös, J. H. Spencer 〔3〕 and P. J. Lorimer, P. R. Mullins 〔4〕, respectively. In this paper, the Ramsey numbers $r(mG, nK_3)$ are determined for all graphs offorder 4 with no isolated vertex. The main results we shall prove are

Theorem 1. $r(m2K_2, nK_3) = \begin{cases} 4m+2n-1 & n\leqslant 2m \\ 2m+3n-1 & 2m\leqslant n \end{cases}$

Theorem 2. $r(mK_{1,3}, nK_3) = \begin{cases} 4m+3 & n=1 \\ 4m+4 & n=2 \\ 4m+2n-1 & 3\leqslant n\leqslant 3m \\ m+3n-1 & 3m\leqslant n \end{cases}$

Theorem 3. $r(mP_4, nK_3) = r(mC_4, nK_3) = \begin{cases} 4m+3 & n=1 \\ 4m+2n & 2\leqslant n\leqslant 2m+1 \\ 2m+3n-1 & 2m+1\leqslant n \end{cases}$

Theorem 4. $r(m(K_{1,3}+x), nK_3) = r(m(K_4-x), nK_3) = \begin{cases} 4m+3 & n=1 \\ 4m+2n & 2\leqslant n\leqslant 2m \\ 2m+3n & 2m\leqslant n \end{cases}$

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The proofs of the theorems by induction on m and n are based on the initial values $r(2K_2, K_3)=5$, $r(P_4, K_3)=r(C_4, K_3)=r(K_{1,3}, K_3)=r(K_{1,3}+x, K_3)=r(K_4-x, K_3)=7$, which can be found in [2], $r(K_4-x, 2K_3)=8$, $r(C_4, 3K_3)=10$, $r(K_{1,3}, 3K_3)=9$. The final three values can be proved straightforward simply by considering all the different ways in which the relevant complete graph can be colored. It seems unnecessary to go into details here.

2. Lower Bounds

In this section, we shall construct Ramsey graphs $RG(mG, nK_3)$ for all graphs G of order 4 with no isolated vertex, so that the numbers given are lower bounds for the appropriate Ramsey numbers. In each case this involves describing a graph with one fewer vertices than the number given and having no red mG or green nK_3.

(1) Lower bounds for $r(m2K_2, nK_3)$

For $n\leqslant 2m$, let G_1 be a red K_{4m-1}, G_2 a green K_{2n-1}, and suppose G_1 and G_2 are disjoint. Join each vertex of G_1 to each vertex of G_2 by a green edge. The resulting complete graph has $4m+2n-2$ vertices but no subgraph $m2K_2$ colored red and no subgraph nK_3 colored green. Hence $r(m2K_2, nK_3)\geqslant 4m+2n-1$, $n\leqslant 2m$.

For $2m\leqslant n$, let G_1 be a red K_{2m-1}, G_2 a green K_{3n-1}, and suppose G_1 and G_2 are disjoint. Join each vertex of G_1 to each vertex of G_2 by a red edge. The result is a complete graph with $2m+3n-2$ vertices which has neither red $m2K_2$ nor green nK_3 as a subgraph. We so have $r(m2K_2, nK_3)\geqslant 2m+3n-1$, $2m\leqslant n$.

(2) Lower bounds for $r(mK_{1,3}, nK_3)$

Let $n=1$, for each m, the graph, constructed by joining each vertex of a complete graph G_1 with $4m-1$ vertices and all the edges colored red and each vertex of a disjoint red triangle G_2 by a green edge, contains no red $mK_{1,3}$ and no green K_3, since the green subgraph is a complete bipartite graph $K_{4m-1,3}$ with no odd cycle. Hence $r(mK_{1,3}, K_3)\geqslant 4m+3$.

For $n=2$ and each m, let G be the graph constructed above for $n=1$, v a vertex not in G. Join each vertex of G and v by a green edge. We then have a graph with no red $mK_{1,3}$ and all green trriangles having the vertex v. Hence there exists no green $2K_3$ and $r(mK_{1,3}, 2K_3)\geqslant 4m+4$.

For $3\leqslant n\leqslant 3m$, let G_1 be a red complete graph of order $4m-1$, G_2 a green complete graph of order $2n-1$. Suppose the two graphs are disjoint. Join each vertex of G_1 to each vertex of G_2 by a green edge. The result is a complete graph with $4m+2n-2$ vertices, which has no subgraph $mK_{1,3}$ colored red and no subgraph nK_3 colored green. Therefore, $r(mK_{1,3}, nK_3)\geqslant 4m+2n-1$, $3\leqslant n\leqslant 3m$.

For $3m\leqslant n$, let G be a graph constructed by joining each vertex of a green K_{3n-1} and each vertex of a red K_{m-1} by a red edge. It is clear that G contains neither red $mK_{1,3}$ nor green nK_3 as a subgraph, as desired.

(3) Lower bounds for $r(mP_4, nK_3)$, $r(mC_4, nK_3)$, $r(m(K_{1,3}+x), nK_3)$ and $r(m(K_4-x), nK_3)$

For $n=1$, let G_1 be a red K_{4m-1}, G_2 a red K_3 and suppose they are disjoint. Join each vertex of G_1 and each vertex of G_2 by a green edge. Since the green subgraph of the resulting graph is bipartite, there is no green triangle. Moreover, over, it has no red mP_4, so $r(mP_4, K_3)\geqslant 4m+3$. Note that $P_4\subset C_4\subset K_4-x$, $P_4\subset K_{1,3}+x$, hence $r(mG, K_3)\geqslant 4m+3$, where G may be C_4, $K_{1,3}+x$, K_4-x.

For $2\leqslant n\leqslant 2m+1$, let G_1 be a red K_{4m-1}, G_2 a green K_{2n-1}, G_3 a vertex, and suppose the three graphs are disjoint. Join each vertex of G_1 and each vertex of G_2 and G_3 by a green edge, and each vertex of G_2 and the vertex of G_3 by a green edge. The result is complete graph with $4m+2n-1$ vertices which contains no green nK_3 and no red mP_4. It follows that

$$r(mC_4, nK_3)\geqslant r(mP_4, nK_3)\geqslant 4m+2n, \qquad 2\leqslant n\leqslant 2m+1;$$
$$r(m(K_4-x), nK_3)\geqslant r(m(K_{1,3}+x), nK_3)\geqslant 4m+2n, \qquad 2\leqslant n\leqslant 2m.$$

For $2m+1\leqslant n$, it follows from $r(m2K_2, nK_3)\geqslant 2m+3n-1$ that $r(mC_4, nK_3)\geqslant r(mP_4, nK_3)\geqslant 2m+3n-1$ by $mC_4\subset mP_4\subset m2K_2$.

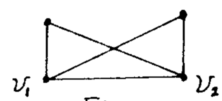
Finally, for $2m\leqslant n$, let G_1 be a red K_{2m-1}, G_2 a green K_{3n-1}, G_3 a vertex, and suppose the three graphs are disjoint. Join each vertex of G_2 and each vertex of G_1 and G_3 by a red edge. The resulting complete graph has $2m+3n-1$ vertices which has no green nK_3 and no red $m(K_{1,3}+x)$. Hence $r(m(K_4-x), nK_3)\geqslant r(m(K_{1,3}+x), nK_3)\geqslant 2m+3n$, $2m\leqslant n$.

3. Some Useful Lemmas

Before turning to proving that the numbers given are also the upper bounds for the Ramsey numbers, let us show several useful lemmas which will be used in induction nextssection. The subgraphs whose existence are guaranteed by these lemmas play the same role as the "bowtie" in [3].

Lemma 1. If two-colored complete graph G contains mutually disjoint red K_4-x and green K_3, then it contains a subgraph H_0 of order 6 having a red K_4-x and a green K_3.

Proof. Assume the contrary. Then no vertex of the green K_3 is joined to the vertices $\{v_1, v_2\}$ of K_4-x, shown in Fig., by two red edges, and so each is joined by at least one green edge to $\{v_1, v_2\}$. Hence there are at least three green edges joining $\{v_1, v_2\}$ and the green K_3, so that one of the $\{v_1, v_2\}$ has two green edges joining it to the green triangle. This gives a red K_4-x and a green K_3 having a vertex in common, as required. ∎



Fig

Corollary. If two-colored complete graph G contains mutually disjoint red

K_4-x and green $2K_3$, then it contains a subgraph H_1 of order 8 having a red K_4-x and a green $2K_3$.

Proof. Assume the contrary. With Lemma 1 we may suppose that G has a subgraph H_0 of order 6 as above, where v is a common vertex of red K_4-x and green K_3. Moreover, suppose v_1, v_2 are the vertices of H_0 which G [{v, v_1, v_2}] is a red triangle. Then no vertex of another K_3, whose vertices are u_1, u_2, u_3, is joined to {v_1, v_2} by two red edges. As in the proof of Lemma 1, one of the {v_1, v_2} must be joined to two of {u_1, u_2, u_3}, say u_1 any u_2, by green edges. Hence the subgraph induced by the vertices of H_0 and u_1, u_2 is disired.

Using the same method, we can prove the following

Lemma 2. If two-colored complete graph G contains mutually disjoint red C_4 and green $2K_3$, then it contains a subgraph H_2 of order 8 having a red C_4 and a green $2K_3$.

Lemma 3. If two-colored complete graph G contains mutually disjoint red $2K_2$ and green K_3, then it has a subgraph H_3 of order 6 with a red $2K_2$ and a green K_3.

Lemma 4. If two-colored complete green G contains mutually disjoint red $K_{1,3}$ and a green kK_3, then it has a subgraph with $4+2k$ vertices containing a red $K_{1,3}$ and a green kK_3, where $k=1,2,3$.

A well known result [4] is that

Lemma 5. If $m\geqslant 2$, then $r(mG, H)\leqslant r((m-1)G, H)+p(G)$, where $p(G)$ is the order of G.

Lemma 6. Suppose that any two-colored complete graph containing mutually disjoint red G and green kH contains a subgraph G_0 with $p(G_0)$ vertices which has a red G and green kH. Then if $m\geqslant 1$, $n\geqslant k\geqslant 1$, $r((m+1)G, (n+k)H)\leqslant r(mG, nH)+p(G_0)$.

Proof. Let $p=r(mG, nH)+p(G_0)$. Consider two-colored K_p with no green $(n+k)H$. Since $r(G,(n+k)H)\leqslant r(mG, nH)+kp(H)\leqslant r(mG, nH)+p(G_0)$, K_p contains a red G. Note that $r(mG, kH)\leqslant r(mG, nH)\leqslant r(mG, nH)+p(G_0)-p(G)$, the subgraph formed by subtracting the $p(G)$ vertices of red G has either a red mG or a green kH. In the former case, the desired result follows immediately, and in the latter case, it can be deduced from the hypothesis that K_p has subgraph G_0 containing a red G and a green kH. Removing these $p(G_0)$ vertices we get a subgraph G_1 which has $r(mG, nH)$ vertices. Hence G_1 has either a red mG or a green nH. The latter is excluded and the former again can be adjoined to the G in G_0 to give a red $(m+1)G$.

$$r((m+1)G,(n+k)H)\leqslant p=r(mG, nH)+p(G_0).$$

4. Upper Bounds

We finally show that the numbers given are upper bounds for the Ramsey numbers in all cases and complete the proofs of our main results. The upper bounds are derived by induction based on the initial values given in Sec. 1, and with induction steps by the lemmas proved in Sec. 3.

(1) Upper bounds for $r(m(K_4-x), nK_3)$

First, since $r((K_4-x), K_3)=7$ and $r(K_4-x, 2K_3)=8$, it follows from Lemma 5 by induction that $r(m(K_4-x), K_3)\leqslant 4m+3$ and $r(K_4-x, nK_3)\leqslant 2+3n$ for $n\geqslant 2$.

Suppose that for $m>2$, $n\geqslant 2$,

$$r((m-1)(K_4-x), nK_3)\leqslant\begin{cases}4(m-1)+2n & 2\leqslant n\leqslant 2(m-1)\\ 2(m-1)+3n & 2(m-1)\leqslant n.\end{cases}$$

Then by Lemma 5, we have $r(m(K_4-x), nK_3)\leqslant 4m+2n$, $2\leqslant n\leqslant 2(m-1)$. The hypotheses of Lemma 6 have been shown, by Lemma 1 and its corollary, to be satisfied for $n=2m-1$ and 2m, respectively. Hence

$r(m(K_4-x),(2m-1)K_3)\leqslant r((m-1)(K_4-x),2(m-1)K_3)+6\leqslant 4(m-1)+2\ 2(m-1)+6$ $=4m+2(2m-1)$ and $r(m(K_4-x), 2mK_3)\leqslant r((m-1)(K_4-x),(2m-2)K_3)+8\leqslant 4m+2{*}2m$

The result $r(m(K_4-x), nK_3)\leqslant 2m+3n$, $2m\leqslant n$, follows by induction on n from Lemma 5 based on the initial condition just proved $r(m(K_4-x), 2mK_3)\leqslant 4m+2{*}2m$ $=2m+3{*}2m$.

Since $m(K_{1,3}+x)\subset m(K_4-x)$,

$$r(m(K_{1,3}+x), nK_3)\leqslant r(m(K_4-x), nK_3)\leqslant\begin{cases}4m+3 & n=1\\ 4m+2n & 2\leqslant n\leqslant 2m\\ 2m+3n & 2m\leqslant n\end{cases}$$

Theorem 4 holds.

(2) Upper bounds for $r(mC_4, nK_3)$

By $C_4\subset K_4-x$, $r(mC_4, nK_3)\leqslant\begin{cases}4m+3 & n=1\\ 4m+2n & 2\leqslant n\leqslant 2m\end{cases}$

Since $r(C_4, 3K_3)=10$, $r(C_4, nK_3)\leqslant 3n+1$, $n\geqslant 3$.

Now assume that for some $m>1$, the result $r(mC_4, nK_3)\leqslant 2m+3n-1$, $2m+1\leqslant n$, has been proved. Then by Lemma 2 and Lemma 6, $r((m+1)C_4,(2m+3)K_3)\leqslant r(mC_4,(2m+1)K_3)+8\leqslant 2m+(6m+3)-1+8=2(m+1)+3n-1$, $n=2m+3$. It follows by induction on n, $n\geqslant 2m+3$, from Lemma 5 that $r((m+1)C_4, nK_3)\leqslant 2(m+1)+3n-1$. Hence, since $P_4\subset C_4$,

$$r(mP_4, nK_3)\leqslant r(mC_4, nK_3)\leqslant\begin{cases}4m+3 & n=1\\ 4m+2n & 2\leqslant n\leqslant 2m+1\\ 2m+3n-1 & 2m+1\leqslant n\end{cases}$$

The proof of Theorem 3 is completed.

(3) Upper bounds for $r(m2K_2, nK_3)$

By [3], Theorem 9, $r(m2K_2, 2mK_3)=8m-1$. Hence $r(m2K_2, nK_3)\leqslant 2m+3n-1$.

$n \geqslant 2m$, since $2K_2 \subset C_4$.

For n even, $r((n/2)2K_2, nK_3) = 4(n/2) + 2n - 1$. Since $r(m2K_2, nK_3) \leqslant r((m-1)2K_2, nK_3) + 4$, $m \geqslant 2$, it follows by induction on m that $r(m2K_2, nK_3) \leqslant 4m + 2n - 1$, $m \geqslant (n/2)$.

For $n > 1$ odd, by Lemma 3 and Lemma 6, we have $r(((n+1)/2)2K_2, nK_3) \leqslant r(((n-1)/2)2K_2, (n-1)K_3) + 6 \leqslant 4 * ((n+1)/2) + 2n - 1$. The desired result $r(m2K_2, nK_3) \leqslant 4m + 2n - 1$, $m \geqslant (n+1)/2$, will follow from the initial condition just derived by induction on m immediately. This proves Theorem 1.

(4) Upper bounds for $r(mK_{1,3}, K_3)$

The remaining case we now consider is that for $K_{1,3}$, which will also conclude our proof.

$K_{1,3} \subset K_4 - x$ implies that $r(mK_{1,3}, K_3) \leqslant 4m + 3$ and $r(mK_{1,3}, 2K_3) \leqslant 4m + 4$. Based on the initial value $r(K_{1,3}, 3K_3) = 9$, it is given by induction and Lemma 5 that $r(K_{1,3}, nK_3) \leqslant 3n$, $n \geqslant 3$.

Suppose for $m \geqslant 1$, $n \geqslant 3$, $r(mK_{1,3}, nK_3) \leqslant \begin{cases} 4m + 2n - 1 & 3 \leqslant n \leqslant 3m \\ m + 3n - 1 & 3m \leqslant n \end{cases}$

Then $r((m+1)K_{1,3}, nK_3) \leqslant 4(m+1) + 2n - 1$, $3 \leqslant n \leqslant 3m$. By Lemma 4 and Lemma 6, we have $r((m+1)K_{1,3}, (3m+k)K_3) \leqslant r(mK_{1,3}, 3mK_3) + 4 + 2k \leqslant 4(m+1) + 2(3m+k) - 1$, where $k = 1, 2, 3$.

Hence Lemma 4 anh Lemma 6 and the above result enable us to prove by induction that $r((m+1)K_{1,3}, nK_3) \leqslant (m+1) + 3n - 1$, where $n \geqslant 3m$. Theorem 2 is proved.

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