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Cardinality of a Class of (0, 1) Matrices Covering a Given Matrix*

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Let R and S be two vectors whose components are m and non-negative integers, respectively. Let P be an $m\times n$ $(0,1)$-matrix with column sums at most one. Let $\mathfrak{A}_P$ (R, S) be the class consisting of all $m\times n$ $(0,1)$-matrices with row sum vector R and columu sum vector S, which cover P. In this paper we derive a lower bound to the cardinality of class $\mathfrak{A}_P$ (R, S), which can be computed readily.

Let $R=(r_1, r_2, \cdots, r_m)$ and $S=(s_1, s_2, \cdots, s_n)$ be vectors with nonnegative integral entries and $r_1+\cdots+r_m=s_1+\cdots+s_n$. Let $\mathfrak{A}$ (R, S) denote the class of all $m\times n$ $(0,1)$-matrices with i^{th} row sum r_i and j^{th} columu sum s_j for $1\le i\le m$ and $1\le j\le n$.

Let $A=(a_{ij})$ and $B=(b_{ij})$. Then we say $A\ge B$ or A covers B if and only if $a_{ij}\ge b_{ij}$ for all pair i, j. Let P be an $m\times n$ $(0,1)$-matrix with j^{th} column sum ε_j ($\varepsilon_j=0$ or 1) for $1\le i\le n$. Define

$$\mathfrak{A}_P(R, S) = \{A\in\mathfrak{A}(R, S) \mid A\ge P\}.$$

So that the class $\mathfrak{A}_P$ (R, S) is not trivially empty, we require that no row (column) sum of P is greater than the corresponding row (column) sum in R (S). Let A^* be a matrix with row sums $r_1, r_2, \cdots r_m$. Also, A^* has $1'$s wherever P has $1'$s. The remaining $1'$s in A^* are placed asfarto the left as possible. Define S^* to be the j^{th} columu sum of A^*. We call vector $S^*=(s_1^*, s_2^*, \cdots s_n^*)$ the P-required conjugate of the vector $R=(r_1, \cdots, r_m)$. Anstee proved that $\mathfrak{A}_P$ (R, S) is non-empty if and only if

$$\sum_{1\le i\le t} s_i^* \ge \sum_{1\le i\le t} s_i \qquad (1\le t\le n), \tag{1}$$

where $s_1\ge s_2\ge\cdots\ge s_n$. A more difficult problem is to determine the card-

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inality of the class $\mathfrak{A}_{\mathrm{P}}(\mathrm{R}, \mathrm{S})$. In this paper we derive a lower bound on $|\mathfrak{A}_{\mathrm{P}}(\mathrm{R}, \mathrm{S})|$.

Let $S' = (s_1', s_2', \cdots, s_n')$ and $S'' = (s_1'', s_2'', \cdots, s_n'')$ be two vectors with non-negative integral entries. we say that S'' is weakly majorized by S', and write

$$S'' \prec S', \tag{2}$$

if

$$\sum_{1\le i\le t} S_i' \ge \sum_{1\le i\le t} S_i'' \ (1\le t\le n-1), \quad \sum_{1\le i\le n} S_i' = \sum_{1\le i\le n} S_i'' . \tag{3}$$

Now suppose that $S'' \prec S'$, $S'' \ne S'$ and

$$s_1'' \ge s_2'' \ge \cdots \ge s_n'' . \tag{4}$$

Let i_p $(1\le p\le q)$ be all the subscripts for which

$$s_{i_p}'' \ge s_{i_p}', \ 1\le p\le q . \tag{5}$$

Let k be the largest subscript smaller than i_1 such that $s_k' > s_k''$. According to (3) and (4), k exists. Obviously,

$$s_k' > s_k'', \ s_j'' = s_j', \ k < j < i_1 . \tag{6}$$

Let $d_{ki_1} = \min(s_k' - s_k'', s_{i_1}'' - s_{i_1}')$ and $S^{(1)} = (s_i^{(1)})$, where $s_i^{(1)} = s_i$ $(1\le i\le n)$ except that $s_k^{(1)} = s_k' - d_{ki_1}$ and $s_{i_1}^{(1)} = s_{i_1}' + d_{ki_1}$. Then

$$S'' \prec S \prec S' .$$

Moreover, the number of corresponding components in S'' and $S^{(1)}$ which are equal is at least one more than those which are equal in S'' and S'. We can obtain in a similar fashion a vector $S^{(2)}$ such that

$$S'' \prec S^{(2)} \prec S^{(1)},$$

and the number of corresponding components in S'' and $S^{(2)}$ which are equal is at least one more that those which are equal in S'' and $S^{(1)}$. Repeating this process, we obtain a sequence of vectors:

$$S'' = S^{(t)} \prec S^{(t-1)} \prec \cdots \prec S^{(2)} \prec S^{(1)} \prec S^{(0)} = S' . \tag{7}$$

The sequence of vectors is called the total chain from S' to S'', whenever S'' satisfies (4). If $q = 0$ we have $t = 0$ in (7).

As mentioned above the reader may also refer to [2] — [5].

We define ω_{P}, which is a function of S' and S'' satisfying (1) —— (6), as follows:

$$\omega_{\mathrm{P}}(S'', S') = \binom{s_k' - s_{i_1}' - \varepsilon_k}{d_{ki_1}}$$

Theorem Let P be an $m \times n$ (0, 1) – matrix with i^{th} column sum ε_i ($\varepsilon_i = 0$ or 1, for $1 \leq i \leq n$. If (1) holds, and $s_1 \geq s_2 \geq \cdots \geq s_n$, then

$$|\mathfrak{A}_P(\mathbf{R},\ \mathbf{S})| \geq \prod_{0 \leq i \leq n-1} \omega_P(\mathbf{S},\ \mathbf{S}^{(i)}) \geq 1, \tag{8}$$

where $\mathbf{S}^{(i)}$ $(0 \leq i \leq t-1)$ are all the elements except $\mathbf{S}^{(t)}$ in the total chain from vector $\mathbf{S}^*$ to $\mathbf{S}$.

Proof Let $\mathbf{S}'' \prec \mathbf{S}'$ be two vectors satisfying (4) — (6) Let $\mathbf{A}' \in \mathfrak{A}_P(\mathbf{R},\ \mathbf{S}')$, and $\mathbf{A}'_{ki_1}$ be the $m \times 2$ matrix consisting of the k^{th} column $\mathbf{A}'_k$ and the $i_1'^{th}$ column $\mathbf{A}'_{i_1}$ of $\mathbf{A}'$. We define a 1 in an $m \times n$ $(0,1)$-matrix to be free if it is not in the same position as a 1 in $\mathbf{P}$.

There are at least $s'_k - s'_{i_1} - \varepsilon_k$ rows in $\mathbf{A}'_{ki}$ that are of the form $(1,0)$, in which the 1 is free. We exchange the elements in each of any d_{ki_1} rows in $\mathbf{A}'_{ki_1}$ that are of the form $(1,0)$, in which the 1 is free, such that at least one of the column sums of the resulting matrix $\mathbf{A}''_{ki_1}$ is equal to the corresponding component of the vector $(s''_k,\ s''_{i_1})$.

By (5) and (6), we have

$$s'_k - s'_{i_1} - \varepsilon_k \geq d_{ki_1}.$$

The number of ways to select d_{ki_1} rows in $\mathbf{A}'_{ki_1}$ that are of the form $(1,0)$, in which the 1 is free, is at least $\omega_P(\mathbf{S}'',\ \mathbf{S}')$, we change those elements in the k^{th} and $i_1'^{th}$ columns in $\mathbf{A}'$ as we did in $\mathbf{A}'_{ki_1}$, and do not change other elements. Thus, we obtain a set of matrices which are different from each other, and the number of the matrices is at least $\omega_P(\mathbf{S}'',\ \mathbf{S}')$.

We apply the above method for $\mathbf{S} \prec \mathbf{S}^*$ at first, and then for $\mathbf{S} \prec \mathbf{S}^{(1)}$, $\cdots$, finally for $\mathbf{S} \prec \mathbf{S}^{(t-1)}$. Let the λ^{th} and the μ^{th} component of $\mathbf{S}^{(l-1)}$ be changed when $\mathbf{S}^{(l-1)}$ are changed into $\mathbf{S}^{(l)}$ $(\lambda < \mu)$, and the ξ^{th} and the η^{th} component of $\mathbf{S}^{(l)}$ when $\mathbf{S}^{(l)}$ are changed into $\mathbf{S}^{(l+1)}$ $(\xi < \eta)$. Then

$$\xi \leq \lambda < \mu \leq \eta, \tag{9}$$

and at most one of equalities holds. If $\mathbf{D}$ is a matrix, we denote its h^{th} column by $\mathbf{D}_h$. Now let $\mathbf{A}$ and $\mathbf{B}$ be two distinct intermediate matrices with j^{th} column sum $s_j^{(l)}$ for $1 \leq j \leq n$. Then there exist e and f: $\lambda \leq e < f \leq \mu$, such that (see the lemma in [1])

$$\mathbf{A}_e \neq \mathbf{B}_e,\ \mathbf{A}_f \neq \mathbf{B}_f.$$

Thus the matrices with j^{th} column sum $s_j^{(l+1)}$ for $1 \leq j \leq n$, which are obtained by $\mathbf{A}$, are different from those by $\mathbf{B}$. Therefore, the resulting matrices are all distinct, and (8) is true.

Simply take $\mathbf{P} = 0$ and note that $\overline{\mathbf{S}} = \mathbf{S}^*$, our theorem reduces to Wei's therom [3].

References

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