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Numbers of Functional Lattices*

Yang An-zhou
(The Industrial University of Peking)

Definion 1. Let L be a Complete lattice, X be an infinite set, $L^X=\{f: f$ is a function & domain $f=X$ & range $f\subseteq L\}$, Define operations of lattice $\vee$, $\wedge$: $(f\vee g)(x)=f(x)\vee g(x)$, $(f\wedge g)(x)=f(x)\wedge g(x)$, $(\vee\{f_i: i\in I\})(x)=\vee\{f_i(x): i\in I\}$, $(\wedge\{f_i: i\in I\})(x)=\wedge\{f_i(x): i\in I\}$.

Defmition 2. $\tau(\subseteq L^X)$ is a functional lattice (or a $\vee$-closed and $\wedge$-finite closed lattice) iff $(\vee\Sigma\vee a(a\in\Sigma\&f_a\in\tau)\to\vee\{f_a: a\in\Sigma\}\in\tau)\&(\forall f, g\vee\tau)(f\wedge g\in\tau)\&(((\forall x\in X)(f_0(x)\to 0))\to(f_0\in\tau))\&(((\forall x\in X)(f_1(x)=1))\to(f_1\in\tau))$. $\mathbf{C}=\{\tau: \tau\subseteq L^X$ & τ is a functional lattice$\}$. $\tau(\subseteq L^X)$ is a complete lattice of functions iff (τ is a functional lattice & $((\forall\Sigma\,\forall a(a\in\Sigma$ & $f_a\in\tau))\to(\wedge\{f_a\cdot a\in\Sigma\}\in\tau)))$, $\mathbf{K}=\{\tau: \tau\subseteq L^X$ & τ is a complete lattice of functions$\}$.

Definition 3. For $\tau, \sigma\in\mathbf{C}$, define $\tau\cong\sigma$ iff $\exists\varphi(\varphi$ is a lattice-isomorph from τ to $\sigma)$, $\hat{\tau}=\{\sigma: \sigma\in\mathbf{C}\ \&\ \sigma\cong\tau\}$; $\tau\sim\sigma$ iff $\exists\varphi(\varphi$ is a bijection on X & φ is continuous & φ^{-1} is continuous), where φ is continuous iff $((\forall g\in\sigma)(\varphi^{-1}(g)\in\tau))$, $(\forall x\in X)(\varphi^{-1}(g))(x)=g(\varphi(x)))$, φ^{-1} is continuous iff $((\forall f\in\tau)(\varphi(f)\in\sigma))$, $(\forall x\in X)\left((\varphi(f))(x)=\begin{cases}\vee\{f(z): z\in\varphi^{-1}[\{y\}]\}, & \text{where } \varphi^{-1}[\{y\}]\neq\phi(\text{empty})\\ 0\end{cases}\right)$, $\langle\tau\rangle=\{\sigma: \sigma\in\mathbf{C}\ \&\ \sigma\sim\tau\}$. $\mathbf{C}_1=\{\hat{\tau}: \tau\in\mathbf{C}\}$, $\mathbf{C}_2=\{\langle\tau\rangle: \tau\in\mathbf{C}\}$, $\mathbf{K}_1=\{\hat{\tau}: \tau\in\mathbf{K}\}$, $\mathbf{K}_2=\{\langle\tau\rangle: \tau\in\mathbf{K}\}$. $[0,1]=\{x: x$ is a real number $\&\ 0\leq x\leq 1\}$.

Theorem 1—3. If $2\leq|L|\leq 2^{|X|}$, then $|\mathbf{C}|=|\mathbf{C}_1|=|\mathbf{C}_2|=\exp(\exp(|X|))=2^{2^{|X|}}$.

Theorem 4—6. If $2\leq|L|\leq|X|$ and $(\exists Q\subset L)(\forall l\in L)(\exists H\subset Q)(\vee H=\sup H=l)$, then $|\mathbf{K}|=|\mathbf{K}_1|=|\mathbf{K}_2|=\exp(|X|)=2^{|X|}$.

Theorem 7—9. If $L=[0, 1]$, then $|\mathbf{C}|=|\mathbf{C}_1|=|\mathbf{C}_2|=\exp(\exp(|X|))=$ $=2^{2^{|X|}}$

Theorem 10—12. If $L=[0, 1]$, then $|\mathbf{K}|=|\mathbf{K}_1|=|\mathbf{K}_2|=\exp(|X|)=2^{|X|}$.

References

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