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Surjectivity of Generaized Expanding Maps and Some Characterizations of Monotone Operators*

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In [1] L. Nirenberg stated the following problem: Suppose T is a continuous map $H \to H$ (H is a Hilbert space), which is expanding, i.e. $\|Tx-Ty\| \geqq \|x-y\|$, and $T(0)=0$. Suppose T map a neighborhood of the origin onto a neighborhood of the origin. Does T maps H onto H? In [2] Kung-Ching Chang and Li Shujie answered positively the problem when T is differentiable.

In this paper we discuss more general cese. we give some surjectivity theorems for generalized expanding maps and some characterizations of monotone operators.

Definition 1. Suppose X, Y are two Banach spaces, T is a continuous map $X \to Y$, $c(r)$ is a monotone non-increase positive function, then we say that T is a generalized expanding map, if T satisfies

$$\|Tx-Ty\| \geqq c(\max(\|x\|, \|y\|))\|x-y\| \ \forall x, y \in X$$

and
$$\int_1^{\infty} c(r)\,dr = +\infty.$$

Obviously, an expanding map must be a generalized expanding map.

Theorem 1 Suppose T is a generalized expanding map from a Banach space X into a Banach space Y. T is Frechet-differentiable in X and $\forall x_0 \in X$, we have $\overline{\lim_{x \to x_0}} \|T'(x)-T'(x_0)\| < c(\|x_0\|)$, $T(0)=0$. T maps a neighborhood of the origin of X onto a neighborhood of the origin Y, then $TX=Y$.

Remark 1. Obviously, this theorem contains theorem 1 in [1].

We need a following proposition, see [3].

Proposition 1. Suppose (1) k is a bounded linear operator from a

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Banach space X onto a Banach space Y, k possesses bounded inverse k^{-1}, (2) G is a continuous map from an open Ball B $(0, r)$ in X into Y, $G(0)=0$, such that

$$\|G(x_1)-G(x_2)-k(x_1-x_2)\| \leqq \delta \|x_1-x_2\|$$

for $\|x_1\| \leqq r$, $\|x_2\| \leqq r$, $M=\|k^{-1}\| \leqq \frac{1}{\delta}$, then $y=G(x)$ have a solution x in B $(0, r)$ for $\|y\|<p=\frac{r(1-M\delta)}{M}$.

Lemma 1. Suppose T is a generalized expanding map from a Banach space X into Banach space Y, T is Frechet differentiable in X. For any $x_0 \in X$ we have

$$\overline{\lim_{x \to x_0}} \|T'(x)-T'(x_0)\| < c(\|x_0\|)$$

then $\forall \eta \in (\overline{\lim_{x \to x_0}} \|T'(x)-T'(x_0)\|, c(\|x_0\|))$ there exists $r>0$ such that $y=Tx$ have a solution x in B (x_0', r) for $\|y-Tx_0\| < \frac{r(1-\|T'(x_0)^{-1}\|\eta)}{\|T'(x_0)^{-1}\|}$

Proof Let's take $k=T'(x_0)$ in proposition 1. By hypothesis we know for any $\eta \in (\overline{\lim_{x \to x_0}} \|T'(x)-T'(x_0)\|, c(\|x_0\|)$ there exists $r>0$ such that

$$\|T'(x)-T'(x_0)\| < \eta, \ \forall x \in B(x_0, r).$$

Let $V(h)=T(h)-T(x_0)-T'(x_0)(h-x_0)$, then we have

$$\|T(h)-T(1)-T'(x_0)(h-1)\| = \|V(h)-V(1)\| \tag{1}$$

for $\|h-x_0\|<r$, $\|1-x_0\|<r$ and

$$V'(h)=T'(h)-T'(x_0) \tag{2}$$

By the mean value theorem, there exists a $\xi \in [0,1]$ such that

$$\|V(h)-V(1)\| \leqq \|V'(h-1)+1)\| \ \|h-1\|. \tag{3}$$

Combining (1), (2), (3), we obtain

$$\|T(h)-T(1)-T'(x_0)(h-1)\| \leqq \|T'(\xi(h-1+1)-T'(x_0)\| \ \|h-1\| \leqq \eta \|h-1\|. \tag{4}$$

Let's take $\delta=\eta$ in proposition 1. From $\|T'(x_0)^{-1}\| \leqq \frac{1}{c(\|x_0\|)}$.(because $\|Tx-Ty\| \geqq c(\max(\|x\|,\|y\|)) \|x-y\|$), so $\|T'(x_0)x\| > c(\|x_0\|)\|x\|$ we have $\frac{1}{\delta}=\frac{1}{\eta} > \frac{1}{c(\|x_0\|)} \geqq \|T'(x_0)^{-1}\|=M$. So we get the lemma from proposition 1.

The proof of theorem 1 Since $T'(0)^{-1}$ exists (see lemma 1 in [1]), $\|T'(0)^{-1}\| \leqq \frac{1}{c(0)}$, $c(0)>0$, and $\overline{\lim_{x \to x_0}} \|T'(x)-T'(x_0)\| < c(\|x_0\|) \leqq c(0)$, so $T'(x)^{-1}$ exists at a neighborhood of the origin of X (see lemma 3 in [1]). For any $x \neq 0$, there exists a positive number $r(x)$, where

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$$r(x)=\sup\{r|[T'(\rho\frac{x}{\|x\|})]^{-1} \text{ exists as } 0<p\leqq r\}.$$

Let $A=\{x=r\cdot y/\|y\| \mid \forall y\in X,\ r<r(y)\}$,

we want to prove $A=X$. If this is not true, then there exists a $\bar{x}\in X$ such that $r(\bar{x})=M_0$, where M_0 is a finite number. Let $\hat{x}=r(\bar{x})^{\bar{x}/\|\bar{x}\|}$. By the hypothesis of theorem, we know that for any $\varepsilon>0$, where $0<\varepsilon<\min\left\{\frac{c(\|\hat{x}\|)}{2},\ c(r(\hat{x}))-\overline{\lim_{x\to x}}\|T'(x)-T'(\hat{x})\|\right\}$, there exists a $\delta>0$ such that $\|T'(x)-T'(\hat{x})\|<\varepsilon<c(\|\hat{x}\|)-\varepsilon=c(r(\bar{x}))-\varepsilon$ as $\|x-\hat{x}\|<\delta$. Since $T'(x_1)^{-1}$ exists and $\|T'(x_1)^{-1}\|<\frac{1}{c(r(\bar{x}))-\varepsilon}$ where $x_1=p\bar{x}/\|\bar{x}\|$, $0<r(\bar{x})-\rho<\delta$, therefore $T'(\hat{x})^{-1}$ exists and $\|T'(\hat{x})^{-1}\|<\frac{1}{c(r(\bar{x}))}$ (see lemma 3 in [1]) and also $T'(x)^{-1}$ exists and is bounded for any $\|x-\hat{x}\|<\delta$ which contradicts $\|\hat{x}\|=M=r(x)$. So $A=X$ and from lemma 1, we know that TX is an open set in Y. Now we want to prove $TX=Y$. The following method is due to F. Browder [4]. Suppose TX is different from Y. Then there exists a point $y\notin TX$ closest to the origin. Let $y(t)$, $0\leqq t\leqq 1$ be a parametrization of an open line segment in TX approaching y, $x(t)=T^{-1}(y(t))$. The function $x(t)$ cannot approach a limit x_1 as $t\to 1$, since for that limit, $T(x_1)=y$ contrary to the assumption that y does not lie in $T(x)$. Furthermore $\|x(t)\|$ cannot remain bounded as $t\to 1$, if it's not true, then there exists a constant $M_1>0$ such that $\|x(t)\|<M_1$ as $0\leqq t\leqq 1$, then from $\|y(t_n)-y(t_m)\|\to 0$ as t_n, $t_m\to 1$, we obtain

$$\|y(t_n)-y(t_m)\|=\|Tx(t_n)-Tx(t_m)\|\geqq c(\max(\|x(t_n)\|,\ \|x(t_m)\|))\cdot\|x(t_n)-x(t_m)\|.$$

Since $c(\max(\|x(t_n)\|,\ \|x(t_m)\|)\geqq c(M_1)>0$, so $\{x(t_n)\}$ is a Cauchy sequence as $t_n\to 1$. Which contradicts that $x(t)$ cannot approach a limit x_1 as $t\to 1$.

$x(t)$ is a continuous function as $t\in[0,1)$. In fact, for any $t_0\in[0,1)$ since $x(t_0)=T^{-1}y(t_0)$, $y(t)\to y(t_0)$ as $t\to t_0$. The continuity of $x(t)$ immediately get from lemma 1. Thus we may find a sequence of parameter values $\{t_k\}$, $t_k<t_{k+1}$, such that $\|x(t_k)\|=k$ for $k\geqq k_0$. Hence $\|x(t_k)-x(t_{k+1})\|\geqq 1$. For this sequence

$$\|y(t_{k+1})-y(t_k)\|\geqq c(k+1)\|x(t_{k+1})-x(t_k)\|\geqq c(k+1).$$

Hence

$$\sum_k c(k)\leqq\sum_k\|y(t_{k+1})-y(t_k)\|<+\infty,$$

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which contradicts the assumption that $\int_1^{\infty} c(r)dr = +\infty$. Thus $AX = Y$ and the theorem is proved.

Now we consider potential operator.

Suppose X is a Banach space, X^* denotes the dual space of X.

Defintion 2 .Suppose T is a mapping from $X \to X^*$. Then we say T is a potential operator if there is a real-valued functional $I(x) \in C^1(X, R)$ such that the Frechet derivative of $I(x)$, $I'(x) = T(x)$.

Suppose X is a Hilbert space.

Theorem2 . Suppose T is a potential generalized expanding map, $T \in C^1(X)$ then T maps X onto X.

Proof: since T is a potential operator iff $T'(x)$ is symmetric $\forall x \in X$, we immediately get that $[T'(x)]^{-1}$ esists $\forall x \in X$. Using the implicit function theorem we know that $T(x)$ is an open mapping from X into X. Obviously, TX is a closed set (see the proof of theorem 1). So $TX = X$. The theorem is proved.

The following theorem is about some characterization of monotone operators.

Theorem3 . Suppose T is a generalized expanding map from a Hilbert space X into X, $T(x) \in C^1(X)$ $\forall x \in X$. $T'(x)$ are symmetric linear operators $\forall x \in X$. If T is a local monotone operator at a neighborhood of the origin of X. Then T must be a monotone operator. Furthermore

$$(TX - Ty,\ x - y) \geqq c(\max(\|x\|,\ \|y\|))\ \|x - y\|^2$$

Proof: From the local monotonicity of T, we know that $T'(0)$ is a monotone operator. Since $\|T'(0)h\| \geqq c(0)\|h\|$, therefore $T'(0)$ is a monotone operator from X onto X (see Nirenberg [2]). $T'(0)^{-1}$ exists and it is bounded operator (Banach theorem) . Hence we obtain that $T'(0)$ is a positive definite operator and $(T'(0)h, h) \geqq c(0)\|h\|^2$ (by $\|T'(0)h\| \geqq c(0)\|h\|$), $(T'(x)h, h) \geq c(\|x\|)\ \|h\|^2$ (by $T \in C^1(X)$ and $\|T'(x)h) \geq c(\|x\|)\ \|h\|$), $\forall x \in X$. So we have $(Tx - Ty, x - y) = (\int_0^1 T'(y + t(x - y))(x - y)dt,\ x - y) \geqq \int_0^1 c(\|y + t(x - y)\|)\|x - y\|^2 dt \geqq c(\max(\|y\|,\ \|x\|)\|x - y\|^2$,

the theorem is proved.

Corollary. Suppose T is generalized expanding potential map from a Hilbert space X into X. If T is a local montone operator at a neighborhood of the origin of X, then T must be a monotone operator. Furthermore $(Tx - Ty, x - y) \geqq c(\max(\|x\|, \|y\|))\ \|x - y\|^2$

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