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Carleson's Proof of A Remarkable Equation*

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The theorem below has a very interesting history worth recounting. In 1964 or 1965 I was working on the boundary for Markov chains during which I found an integral equation which played an important role. It resembles a renewal equation and has the intuitive content of a last exit decomposition. It was not hard to solve it in a weak sense but the strict result suggested by its probabilistic meaning baffled me. Two notes on this equation were published in the *C. R. Acad, Sc.* Paris (communicated by Paul Lévy), tome 260, 1965. The problem has a deceptive simplicity and at least two wrong proofs were sent to me. I gave several lectures about it, including one in Budapest when I visited Rényi in 1967. Eventually the problem was solved by Harry Kesten who turned it into a large question of hitting probabilities of single points (*Memoir of the Amer. Math. Soc.* No. 93, 1969). But his proof is not accessible to people without considerable background in probability theory whereas the original problem was posed as an analytic one. Thus I asked Lennart Carleson whether he could prove it more directly. I recall that he replied at once saying that he was interested in the problem because it resembled something he did in another context. Soon after he sent me a complete and c ncise proof. I spent some time making it a little easier for the average reader and the result is the exposition first published here. Prior to the solution an incorrect one was published in *Zeitschr. für Wahrschein.*, vol. 8, 1967, with acknowledgment of error in vol. 11, 1968. I edited that paper and the referee was S. Watanabe. Neither of us caught the mistake which was due to the failure to check dominated convergence (a most common mistake which has probably infected a non-trivial portion of the mathematical literature). P. A. Meyer found the mistake. After I showed him Carleson's proof in my exposition he had a pupil of his write up a French version of it which was published. However, Carleson as well as I thought that the following version was more readable. In publishing it here I hope that it will give Chinese readers an opportunity to study a simple-looking yet hard problem solved by penetrating analysis, and at the same time to

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appreciate the interaction between various disciplines in mathematics.

Carleson's Proof

Let σ be a decreasing and right continuous function on $(0, \infty)$ such that

$$\sigma(0+)=\infty, \qquad \int_0^1 \sigma(t)\mathrm{d}t<\infty .$$

It is known that there exists a σ-finite measure E on $[0, \infty)$ such that for almost every x (Lebesgue measure) we have

$$\int_0^x \sigma(x-t)\mathrm{d}E(t)=1 . \tag{1}$$

Furthermore for every $x>0$,

$$\int_0^x \sigma(x-t)\mathrm{d}E(t)\leq 1 , \tag{1'}$$

[we shall need only the finiteness of this integral]. It follow from (1) that the measure E is without atom.

Theorem. The equation (1) holds for every $x>0$.

The theorem was proved by Kesten by probability methods. The following proof is due to L. Carleson (March 31, 1969).

Integrating (1) we obtain, if $0<a<b<x_0$:

$$\int_{x_0-b}^{x_0-a}\mathrm{d}x\int_0^x \sigma(x-t)\mathrm{d}E(t)=b-a .$$

Transform the integral into

$$\int_0^{x_0-a}\mathrm{d}E(t)\int_{x_0-b}^{x_0-a}\sigma(x-t)\mathrm{d}x=\int_a^{x_0}-\mathrm{d}E(x_0-y)\int_{x_0-b}^{x_0-a}\sigma(y+x-x_0)\mathrm{d}x .$$

Put

$$\mu(y)=\begin{cases} E(x_0)-E(x_0-y), & \text{for } 0\leq y\leq x_0 , \\ E(x_0), & \text{for } y>x_0 ; \end{cases}$$

and put $\sigma(x)=0$ for $x<0$. We then get

$$\int_0^{\infty}\mathrm{d}\mu(y)\int_a^b \sigma(y-t)\mathrm{d}t=b-a . \tag{2}$$

Thus for every $\delta>0$,

$$1=\int_0^{\infty} d\mu(y)\frac{1}{\delta}\int_0^{\delta}\sigma(y-t)\mathrm{d}t . \tag{2'}$$

For any fixed $\rho>0$, if $y>\rho$, then for sufficiently small δ,

$$\frac{1}{\delta}\int_0^{\delta}\sigma(y-t)\mathrm{d}t\leq\sigma(\rho-\delta)\leq\delta\left(\frac{\rho}{2}\right) .$$

We use this domination to conclude that

$$(3) \qquad \lim_{\eta\downarrow 0}\int_\rho^\infty d\mu(y)\frac{1}{\delta}\int_0^\delta \sigma(y-t)\,dt=\int_\rho^\infty \sigma(y)\,d\mu(y),$$

since σ can be discontinuous only on a countable set to which E assigns zero measure. Therefore if we can prove that *given any* $\varepsilon>0$, *there exists a* $\rho>0$ *such that*

$$(4) \qquad \varlimsup_{\delta\downarrow 0}\int_0^\rho d\mu(y)\frac{1}{\delta}\sigma(y-t)\,dt\le\varepsilon$$

then we shall have from $(2')$ and (3):

$$\int_0\sigma(y)\,d\mu(y)=\lim_{\rho\downarrow 0}\int_0^\rho\sigma(y)\,d\mu(y)=1.$$

This is the desired equation (1) in the new notation, for $x=x_0$. The remainder of the proof is to establish the underscored proposition above.

Note that we have as $\delta\downarrow 0$,

$$\sigma(\delta)\mu(\delta)\le\int_0^\delta\sigma(y)\,d\mu(y)\to 0;$$

also that for every $\delta>0$ and $y>0$:

$$\delta\sigma(\delta)\le\int_0^\delta\sigma(t)\,dt, \qquad \frac{1}{\delta}\int_0^\delta\sigma(y-t)\,dt\le\frac{1}{\delta}\int_0^\delta\sigma(t)\,dt.$$

These relations will be used many times below without comment.

Case 1. Suppose that

$$\varlimsup_{\delta\downarrow 0}\frac{1}{\delta\sigma(\delta)}\int_0^\delta\sigma(t)\,dt<\infty.$$

Then there exists $A<\infty$ such that

$$\int_0^\delta\sigma(t)\,dt\le A\delta\sigma(\delta), \qquad \text{for } \delta\le 1.$$

We have then if $x\le\frac{1}{2}$,

$$x\sigma(x)\le\int_0^x\sigma(t)\,dt\le\int_0^{2x}\sigma(t)\,dt\le A2x\sigma(2x),$$

so that

$$\sigma(x)\le 2A\sigma(2x).$$

For $0<2\delta<\rho<1$, we have

$$\int_0^{2\delta}d\mu(y)\frac{1}{\delta}\int_0^\delta\sigma(y-t)\,dt\le\int_0^{2\delta}d\mu(y)\frac{1}{\delta}\int_0^\delta\sigma(t)\,dt\le\int_0^{2\delta}d\mu(y)A\sigma(\delta)$$
$$\le A\mu(2\delta)\sigma(\delta)\le 2A^2\mu(2\delta)\sigma(2\delta);$$

$$\int_{2\delta}^\rho d\mu(y)\frac{1}{\delta}\int_0^\delta\sigma(y-t)\,dt\le\int_{2\delta}^\rho\sigma\left(\frac{y}{2}\right)d\mu(y)\le 2A\int_{2\delta}^\rho\sigma(y)\,d\mu(y).$$

Hence both integrals can be made $<\varepsilon$ by choosing ρ sufficiently small, and the theorem is proved in t s case.

Case 2.

$$\underline{\lim}_{\delta\downarrow 0}\frac{1}{\delta\sigma(\delta)}\int_0^\delta \sigma(t)\,dt<\infty=\overline{\lim}_{\delta\downarrow 0}\frac{1}{\delta\sigma(\delta)}\int_0^\delta \sigma(t)\,dt.$$

There exist $A<\infty$ and $\delta_i\downarrow 0$ such that for all i:

$$(5)\qquad \int_0^{\delta_i}\sigma(t)\,dt\le A\delta_i\sigma(\delta_i).$$

For an arbitrarily large $B>3A$, put

$$\eta_i=\sup\{\eta:\eta\le\delta_i,\ \frac{1}{\eta\sigma(\eta)}\int_0^\eta\sigma(t)\,dt>B\}.$$

If $0<c<1$, then

$$\frac{1}{c\delta_i\delta(c\delta_i)}\int_0^{c\delta_i}\sigma(t)\,dt\le\frac{1}{c\delta_i\sigma(\delta_i)}\int_0^{\delta_i}\sigma(t)\,dt\le\frac{A}{c};$$

consequently we have $\eta_i\le\frac{A}{B}\delta_i<\frac{\delta_i}{3}$, but $\eta_i>0$ by the hypothesis of the case. By definition we have

$$(6)\qquad \int_0^y\sigma(t)\,dt\le By\sigma(y)\qquad\text{for}\quad \eta_i\le y\le\delta_i;$$

while the right continuity of σ implies

$$(7)\qquad \int_0^{\eta_i}\sigma(t)\,dt=B\eta_i\sigma(\eta_i).$$

It follows at once that

$$\sigma(\eta_i)\le 2\sigma(2\eta_i).$$

Until further notice we shall omit the index i from η_i and δ_i. We have then

$$\int_0^{2\eta}d\mu(y)\frac{1}{\eta}\int_0^\eta\sigma(y-t)\,dt\le\int_0^{2\eta}d\mu(y)B\sigma(\eta)\le 2B\mu(2\eta)\sigma(2\eta).$$

Next, for $2\eta\le y\le\delta$ we have $y/(y-\eta)\le 2$, so that

$$\frac{1}{\eta}\int_0^\eta\sigma(y-t)\,dt\le\sigma(y-\eta)\le\frac{1}{y-\eta}\int_0^{y-\eta}\sigma(t)\,dt\le\frac{y}{y-\eta}\,\frac{1}{y}\int_0^y\sigma(t)\,dt\le 2B\sigma(y)$$

by (6). Hence

$$\int_{2\eta}^\delta d\mu(y)\frac{1}{\eta}\int_0^\eta\sigma(y-t)\,dt\le 2B\int_{2\eta}^\delta\sigma(y)\,d\mu(y).$$

Combining these we have

$$(8)\qquad \int_0^\delta d\mu(y)\frac{1}{\eta}\int_0^\eta\sigma(y-t)\,dt\le 2B\int_0^\delta\sigma(y)\,d\mu(y).$$

Finally, for each i we define two sets as follows:

$C_i=\{y:\delta_i\le y\le\rho;\ \sigma(y-\eta_i)\le 2\sigma(y)\}$, $D_i=\{y:\delta_i\le y\le\rho;\ \sigma(y-\eta_i)>2\sigma(y)\}$;

thus $[\delta_i,\rho]=B_i\cup C_i$. Omitting the index i from C_i, D_i, η_i, δ_i as before, we have since $3\eta\le\delta$,

$$(9)\quad \int_C d\mu(y)\frac{1}{\eta}\int_0^{\eta}\sigma(y-t)\,dt\le\int_C\sigma(y-\eta)\,d\mu(y)\le 2\int_C\sigma(y)\,d\mu(y)\le 2\int_0^{\rho}\sigma(y)\,d\mu(y).$$

Let

$$\lambda_0=\inf D,\qquad \omega_0=[\lambda_0-\eta,\ \lambda_0+\eta],$$
$$\lambda_1=\inf(D\ \omega_0),\qquad \omega_1=[\lambda_1-\eta,\ \lambda_1+\eta],$$
$$\lambda_2=\inf(D\ (\omega_0\cup\omega_1)),\qquad \omega_2=[\lambda_2-\eta,\ \lambda_2+\eta],$$

and so on, until an Index L such that

$$(10)\qquad D\subset\bigcup_{j=0}^{L}\omega_j.$$

Since $\lambda_{j-1}\ge\lambda_i+\eta$, we have $L\le[\frac{\rho}{\eta}]+1$.

Lemma 1. For any $a\ge d>0$, we have

$$\int_a^{a+d}d\mu(y)\le\frac{2d}{\int_0^d\sigma(t)\,dt}.$$

Proof. We have from (2),

$$2d=\int_0^{\infty}d\mu(y)\int_{a-d}^{a+d}\sigma(y-t)\,dt\ge\int_a^{a+d}d\mu(y)\int_{y-a-d}^{y-a+d}\sigma(t)\,dt\ge\int_a^{a+d}d\mu(y)\int_0^d\sigma(t)\,dt.$$

The lemma follows.

Now we have, by (10) and Lemma 1:

$$\int_D d\mu(y)\frac{1}{\eta}\int_0^{\eta}\sigma(y-t)\,dt\le\sum_{j=0}^{L}\int_{\lambda-\eta}^{\lambda_j+\eta}d\mu(y)\sigma(\lambda_j-2\eta)\le\frac{4\eta}{\int_0^{2\eta}\sigma(t)\,dt}\sum_{j=0}^{L}\sigma(\lambda_j-2\eta).$$

Observe that

$$\sigma(\lambda_i-2\eta)\ge\sigma(\lambda_j-\eta)\ge 2\sigma(\lambda_j)\ge 2\sigma(\lambda_{j+2}-2\eta),$$

where the middle inequality follows from the definition of λ_j and the right continuity of σ. Hence in the sum $\sum_{j=0}^{L}$ above, the terms with even [odd] indices from a geometrically decreasing series with ratio $\le 1/2$ and so their sum is bounded by twice its first term. Thus, noting that $\lambda_0\ge 3\eta$,

$$(11)\quad \int_D d\mu(y)\frac{1}{\eta}\int_0^{\eta}\sigma(y-t)\,dt\le\frac{4\eta}{\int_0^{2\eta}\sigma(t)\,dt}2\cdot 2\cdot\sigma(\lambda_0-2\eta)\le\frac{16\eta\sigma(\eta)}{\int_0^{2\eta}\sigma(t)\,dt}.$$

Collecting the estimates (8), (9) and (11), we obtain, restoring the index i and using (7):

$$\int_0^{\rho}d\mu(y)\frac{1}{\eta_i}\int_0^{\eta_i}\sigma(y-t)\,dt\le 2B\int_0^{\delta_i}\sigma(y)\,d\mu(y)+2\int_0^{\rho}\sigma(y)\,d\mu(y)+\frac{16}{B}.$$

Having given $\varepsilon>0$, we choose first B large, the ρ small and finally $i\to\infty(\delta_i\downarrow 0)$ to achieve the required relation (4). Hence the theorem is proved in this case.

Case 3. $\quad\lim_{\delta\downarrow 0}\frac{1}{\delta\sigma(\delta)}\int_0^{\delta}\sigma(t)\,dt=\infty.$

Define the sets C and D as in Case 2 but with η_i and δ_i replaced by η and 3η. The previous estimates (9) and (11) remain valid without any change so that we have

$$\int_{3\eta}^{\rho}\mathrm{d}\mu(y)\frac{1}{\eta}\int_0^{\eta}\sigma(y-t)\mathrm{d}t\leq 2\int_0^{\rho}\sigma(y)\,\mathrm{d}\mu(y)+\frac{16\eta\sigma(\eta)}{\int_0^{2\eta}\sigma(t)\mathrm{d}t}.$$

Under the hypothesis of this case the second term on the right tends to zero as $\eta\downarrow 0$, while the first one can be made $<\varepsilon$ by the choice of ρ. Therefore (4) will be established if we show that

$$\text{(11)}\qquad \varlimsup_{\eta\downarrow 0}\int_0^{3\eta}\mathrm{d}\mu(y)\frac{1}{\eta}\int_0^{\eta}\sigma(y-t)\mathrm{d}t=0.$$

Now the integral above is bounded by

$$\int_0^{3\eta}\mathrm{d}\mu(y)\frac{1}{\eta}\int_0^{\eta}\sigma(t)\mathrm{d}t=\frac{1}{\eta}\int_0^{3\eta}\Sigma(y)\mathrm{d}\mu(y),$$

where

$$\Sigma(y)=\int_0^{y}\sigma(t)\mathrm{d}t.$$

Therefore it remains only to prove the following.

Lemma 2 $\varlimsup_{\eta\downarrow 0}\frac{1}{\eta}\int_0^{\eta}\Sigma(y)\mathrm{d}\mu(y)=0.$

Proof. Write also

$$S(y)=\int_0^{y}\Sigma(t)\mathrm{d}\mu(t).$$

Since σ/Σ is decreasing we may integrate by parts below:

$$1\geq\int_0^{\infty}\sigma(t)\mathrm{d}\mu(t)=\int_0^{\infty}\frac{\sigma(t)}{\Sigma(t)}\Sigma(t)\mathrm{d}\mu(t)\geq -\lim_{t\downarrow 0}\frac{\sigma(t)S(t)}{\Sigma(t)}-\int_0^{\infty}S(t)\mathrm{d}\left(\frac{\sigma(t)}{\Sigma(t)}\right).$$

The last-written limii is zero since

$$\frac{\sigma(t)}{\Sigma(t)}\int_0^{t}\Sigma(s)\mathrm{d}\mu(s)\leq\sigma(t)\mu(t).$$

Now if the lemma were false, then there would exist $\lambda>0$ such that

$$S(\eta)\geq\lambda\eta \qquad \text{for}\quad 0<\eta\leq 1.$$

We should have then

$$-\int_0^{\infty}S(t)\mathrm{d}\left(\frac{\sigma(t)}{\Sigma(t)}\right)\geq-\lambda\int_0^{1}t\mathrm{d}\left(\frac{\sigma(t)}{\Sigma(t)}\right)\geq-\lambda\left\{\frac{\sigma(1)}{\Sigma(1)}-\int_0^{1}\frac{\sigma(t)}{\Sigma(t)}\mathrm{d}t\right\}\geq\frac{\lambda\sigma(1)}{\Sigma(1)}$$
$$+\lambda\log\Sigma(0)=\infty.$$

This is absurd and the lemma is therefore proved.

Remarks We have

$$\lim_{\eta\downarrow 0}\frac{1}{\eta}\int_0^{\eta}[\Sigma(\eta)-\Sigma(y)]\mathrm{d}\mu(y)=\lim_{\eta\downarrow 0}\frac{1}{\eta}\int_0^{\eta}\mathrm{d}y\int_0^{y}\sigma(x)\mathrm{d}\mu(x)=0.$$

Hence Lemma 2 is equivalent to

$$\lim_{\eta \downarrow 0} \frac{\Sigma(\eta)}{\eta} \int_0^{\eta} dE(x) = 0. \tag{12}$$

Proposition The theorem is equivalent to the following analogue of (12):

$$\lim_{\eta \downarrow 0} \frac{\Sigma(\eta)}{\eta} \int_{x_0}^{x_0+\eta} dE(x) = 0, \tag{13}$$

where the lower limit may be replaced by the simple limit.

Proof We have

$$\int_0^{x_0+\eta} \Sigma(x_0+\eta-x)\,dE(x) = x_0+\eta, \quad \int_0^{x_0} \Sigma(x_0-x)\,dE(x) = x_0.$$

Hence

$$\int_0^{x_0} \frac{1}{\eta}\{\Sigma(x_0+\eta-x) - \Sigma(x_0-x)\}\,dE(x) + \frac{1}{\eta}\int_{x_0}^{x_0+\eta} \Sigma(x_0+\eta-x)\,dE(x) = 1. \tag{14}$$

Since

$$\frac{1}{\eta}\{\Sigma(x_0+\eta-x) - \Sigma(x_0-x)\} = \frac{1}{\eta}\int_{x_0-x}^{x_0+\eta-x} \sigma(t)\,dt \leq \sigma(x_0-x),$$

and $\int_0^{x_0} \sigma(x_0-x)\,dE(x) \leq 1$, we have by dominated convergence:

$$1 - \int_0^{x_0} \sigma(x_0-x)\,dE(x) = \lim_{\eta \downarrow 0} \frac{1}{\eta} \int_{x_0}^{x+\eta} \Sigma(x_0+\eta-x)\,dE(x).$$

Is is easy to see that condition (13) is equivalent to the last-written limit being zero.

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