J. Math. Res. & Exposition, 1987, No. 1

Some Combinatorial Problems (I)*

L. C. Hsu (徐利治)

(Institute of Applied Mathematics, Dalian Institute of Technology, Dalian)

In this note and its sequels we will present various unsolved problems mainly occured in our researches of combinatorial analysis.

I. Problems concerning the maximal degree of combinatorial identities

Combinatorial identities are sometimes called binomial coefficient summations in closed form. Two classical books on combinatorial identities, namely [4] and [7], are well-known. As in our previous paper [3], we use the notation $S[p/q, r]$ to denote a combinatorial identity of the type $S: p/q$ containing r independent variables free parameters), where p and q are the numbers of binomial coefficients contained in the numerator and denominator of the summand respectively, and the number r is called the degree of the identity. Thus for instance the following identity (cf. [3])

$$\sum_{k=0}^{\infty} \frac{1}{\binom{k+1+x}{k}\binom{k+1-x}{k}} = \frac{x^2-1}{x^2}\left(1-\frac{\pi x}{\sin(\pi x)}\right) \qquad (x \neq 0)$$

is a formula of type $S[0/2, 1]$, where x is the only free parameter contained so that the degree is one.

For any given p and q (nonnegative integers) with $p+q \geq 1$, define

$$\max\ \text{degree}[p/q] := \sup\{r \mid \exists\, S[p/q, r]\};$$

This is called the maximal degree of identities of the type $S[p/q, \cdot]$.

Problem 1. For given p and q with $p+q \geq 1$, how to determine max degree$[p/q]$? Is it possible to get much closer lower and upper bounds for it?

Problem 2. Does there exist a combinatorial identity of the type $S[3/0, 5]$? If the answer should be negative, how to prove the conjectural result

$$\max\ \text{degree}[3/0] = 4\,?$$

Remark 1. An obvious upper bound for $S[p/q, \cdot]$ is given by $2(p+q)+1$. It is easy to prove the relation max degree$[1/0]$ = max degree$[0/1] = 3$. Moreover, the assertion max degree$[2/0] = 3$ may also be justified without difficulty

* Received Apr. 28, 1986.

Remark 2. There are numerous literature related to combinatorial identities of the type [3/0, 4] (so-called cubic binomial identities). However, it has been shown in our note [3] that various identities of this type obtained by Nanjundiah [4, 6.17], [6], Bizley [2][4, 6.42], krall [4, 6.47], Stanley [4, 6.52] and Andrews [1] respectively are all equivalent, i.e., they are deducible from each other. More recently, Szekely published a note [8] in J. Combin. Theory, Ser. A 40(1985), giving a '5-parameter identity' of the form

$$\binom{a+c+d+e}{a+c}\binom{b+c+d+e}{c+e}=\sum_{k}\binom{a+b+c+d+e-k}{a+b+c+d}\binom{a+d}{k+d}\binom{b+c}{k+c}$$

However, making substitutions $\alpha=a+c$, $\beta=d+e$, $\gamma=b+c$, $\delta=c+e$, this identity can be simplified into the form

$$\binom{\alpha+\beta}{\alpha}\binom{\gamma+\beta}{\delta}=\sum_{k\geq 0}\binom{\alpha+\beta+\gamma-k}{\alpha+\beta+r-\delta}\binom{\alpha+\beta-\delta}{\beta-\delta+k}\binom{\gamma}{k}$$

which is precisely the identity of Krall [4, 6.47]. (Here the original $k+c$ is replaced by k, as the parameter c can be absorbed within the summation). Thus the identity is actually of degeree 4, not of 5.

Remark 3. Takacs' identity [9] has been stated as a corollary in [8], which may be written in the form

$$\sum_{j\geq 0}\binom{\mu_1}{j}\binom{\mu_4-\mu_1}{\mu_2-j}\binom{\mu_2+j}{\mu_4}=\binom{\mu_3}{\mu_4-\mu_2}\binom{\mu_1+\mu_2+\mu_3-\mu_4}{\mu_2}.$$

Actually this is also equivalent to Krall's identity, as may be observed by the nonsingular linear transformation for the parameters

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}=\begin{pmatrix}0&-1&0&1\\ 0&1&1&-1\\ 1&0&0&0\\ 1&0&1&-1\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{pmatrix}$$

with the substitutton $k=\mu_1-j$. Hence in conclusion all the identities of the type S[3/0, 4] mentioned above are equivalent to each other.

In fact, it is known that Andrews' identity and the like can all be derived from the classical Saalschutz identity (cf. E.D.Rainville, "Special Functions", 1960, p. 87)

$${}_3F_2\begin{pmatrix}a, & b, & -n & \\ c, & 1-c+a+b-n & ; & 1\end{pmatrix}=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

where $(c)_0=1$ and $(c)_n=c(c+1)\cdots(c+n-1)$. Thus Saalschutz' formula (first appeared in 1890) should be regarded as the genuine common of various cubic binomial identities. Consequently, it may be a good advice that no more paper on cubic binomial identities should be published hereafter.

Remark 4. As is known, Saalschutz′ identity is a consequence of a certain group representation theory. This suggests that certain classification problems in connection with the structures of combinatorial identities may be better treated form the group representation theoretic view-point.

2. Problems on a kind of asymptotic expansion for binomial sums

Asymptotic evaluation for a particular case of the binomial summation

$$S_n(f)=\sum_{k\geq 0}\binom{n}{f(k)}$$

was treated earlier by Jordan, etc., where f is a certain arithmetic function with $f(k)\uparrow\infty$ $(k\to\infty)$. A more general result has been established in our paper [5]:

Let $f(x)\in C^1([0,\infty))$ and satisfy the following conditions

(i) $f(x)\geq 0$ and $f'(x)\to\infty$ $(x\to\infty)$,

(ii) $f'(x)/f(x)\to 0$ $(x\to\infty)$,

(iii) there exists a positive number $\varepsilon<\frac{1}{10}$ such that $f(x)=o\{(f'(x))^{2-\varepsilon}\}$, $(x\to\infty)$.

Then we have the following three-term asymptotic expansion for n large

$$S_n(f)\sim 2^n\left(\frac{2}{n\pi}\right)^{\frac{1}{2}}\sum_{(m-1)\leq k\leq(m+1)}\exp\left\{\frac{-2}{n}\left(\frac{n}{2}-f(k)\right)^2+O\left(\frac{1}{n^2}\left(\frac{n}{2}-f(k)\right)^3\right)\right\},$$

where $m=[f^{-1}(n/2)]$, f^{-1} being the inverse of f and $[x]$ the integral part of x

However for the case $f(x)=x^r$, one has to assume $r\geq 3$. This leads to the

Problem 3. How to extend the above asymptotic result with a view to including the important particular case $f(x)=x^2$?

Problem 4. Determine what a kind of asymptotic expansion can be obtained for the binomial summation $S_n(f)_q$ in which the summand is the q-binomial coefficient $\left[{n \atop f(k)}\right]_q$ with $0<q\neq 1$.

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