数学研究与评论
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极大的无k个子集两两不相交的子集系的最小容量*

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设S是n元集合，$\mathscr{F}_k(s)$ 是S的子集系，并满足条件：

(1) 对任意$A_1, \cdots, A_k \in \mathscr{F}_k(S)$，存在$A_i$与$A_j$使得$A_i \cap A_j \neq \phi$ ($i \neq j$)，称为无k个子集两两不相交的子集系，或简称为G_{3k}子集系；特别地，又满足：

(2) 若$A_0 \subseteq S$，$A_0 \overline{\in} \mathscr{F}_k(S)$，则存在$A_1, \cdots, A_{k-1} \in \mathscr{F}_k(S)$ 使得$A_i \cap A_j = \phi, 0 \leqslant i \neq j \leqslant k-1$，称为极大的无$k$个子集两两不相交的子集系，或简称为极大的$G_{3k}$子集系.

在〔1〕中已给出了$\mathscr{F}_k(S)$ 的一个上界，但是，$\mathscr{F}_k(S)$ 的下界一直悬而未决. P.Erdös D.Kleitman在〔2〕中问：$\mathscr{F}_k(S)$ 的最小容量等于2^n-2^{n-k}吗？本文的定理1否定了他们的猜想，定理2给出$\mathscr{F}_k(S)$ 的最小容量.

定理1 $\min|\mathscr{F}_k(S)| \leqslant 2^n-2^{n-k+1}$，其中最小运算取遍所有极大的$G_{3k}$子集系.

定理2 $\min|\mathscr{F}_k(S)| = 2^n-2^{n-k+1}$

§1. 定理1的证明

在这一节构造一极大的G_{3k}子集系$\mathscr{F}'_k(S)$，具有：$|\mathscr{F}'_k(S)| = 2^n-2^{n-k+1}$.

设$S=\{1,2,\cdots,n\}$ ($n \geqslant k-1$) 并记$S_1=\{1,\cdots,k-1\}$ 和$S_2=\{k,\cdots,n\}$；又以$\mathscr{B}(S)=\{A: A\subseteq S\}$ 和$\mathscr{B}(S_i)=\{A: A\subseteq S_i\}$ ($i=1,2$) 分别表示S, S_1和S_2上的布尔代数. $\mathscr{B}(S_1)$和$\mathscr{B}(S_2)$ 的积空间被定义为：

$$\mathscr{B}(S_1)\times\mathscr{B}(S_2)=\{(x^{(1)}, x^{(2)}): x^{(i)}\in\mathscr{B}(S_i)\ i=1,2\} \tag{1.1}$$

以$f(A)=(A\cap S_1, A\cap S_2)$ 定义映射$f: \mathscr{B}(S)\to\mathscr{B}(S_1)\times\mathscr{B}(S_2)$. 显然，$f$是$\mathscr{B}(S)$ 与$\mathscr{B}(S_1)\times\mathscr{B}(S_2)$ 之间的序同构映射. 所以，我们可以把$\mathscr{B}(S_1)\times\mathscr{B}(S_2)$ 中的 $(x^{(1)}, x^{(2)})$ 看作$\mathscr{B}(S)$ 中的$x^{(1)}\cup x^{(2)}$，并把$A\cap S_i$ $i=1,2$，叫做A在$\mathscr{B}(S_i)$ 上的投影，记作$P_i(A)$.

令$L(A^{(1)})=\{(A^{(1)}, x^{(2)}): x^{(2)}\in\mathscr{B}(S_2)\}, A^{(1)}\in\mathscr{B}(S_1)$ 和

$$\mathscr{F}'_k(S)=\mathscr{B}(S_1)\times\mathscr{B}(S_2)\setminus L(\phi). \tag{1.2}$$

现在来证：$\mathscr{F}'_k(S)$ 是一极大的G_{3k}子集系.

设$A_1, \cdots, A_k\in\mathscr{F}'_k(S)$，那么$P_1(A_1), \cdots, P_1(A_k)\in\mathscr{B}(S_1)$，由$|S_2|=k-1$，存在$P_1(A_i)$ 与$P_1(A_j)$ 使得$P_1(A_i)\cap P_1(A_j)\neq\phi$，从而获得：$A_i\cap A_j\neq\phi$，故$\mathscr{F}'_k(S)$ 是G_{3k}子集系.

又设$A_0\in\mathscr{B}(S)$，$A_0\overline{\in}\mathscr{F}'_k(S)$. 由 (1.2)，$A_0\in L(\phi)$. 从而，我们取$\{1\},\cdots,\{k-1\}\in\mathscr{F}'_k(S)$，显然，$\{1\},\cdots,\{k-1\}$ 与A_0是两两不相交的，所以，$\mathscr{F}'_k(S)$ 也满足条件 (2).

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此时，我们获得容量为2^n-2^{n-k+1}的极大的G_{3k}子集系$\mathscr{F}_k'(S)$．故定理 1 得证．

§2．定理 2 的证明

要证定理 2，只须证：对任意的极大的G_{3k}子集系$\mathscr{F}_k(S)$ 有

$$|\mathscr{F}_k(S)|\geqslant 2^n-2^{n-k+1}. \tag{2.1}$$

我们对$|S|$进行数学归纳．当$|S|=k-1$时，仅有$\mathscr{F}_k(S)=\mathscr{B}(S)-\{\phi\}$才是极大的$G_{3k}$子集系，显然，

$$|\mathscr{F}_k(S)|=|\mathscr{B}(S)|-1=2^{k-1}-1$$

假设$k-1\leqslant|S|<n$时成立，证$|S|=n$时亦然．设$S=\{1,2,\cdots,n\}$并把 S 分为$S_1=\{1,\cdots,n-1\}$和$S_2=\{n\}$．于是，有$\mathscr{B}(S_1)$与$\mathscr{B}(S_2)$的积空间$\mathscr{B}(S_1)\times\mathscr{B}(S_2)=\{(x^{(1)},x^{(2)})：x^{(i)}\in\mathscr{B}(S_i),i=1,2\}$．令

$$\Gamma(A^{(2)})=\{(x^{(1)},A^{(2)})：x^{(1)}\in\mathscr{B}(S_1)\},\ A^{(2)}\in\mathscr{B}(S_2) \tag{2.2}$$

$$\mathscr{A}=\mathscr{F}_k(S)\cap\Gamma(\phi),\quad \mathscr{B}=\mathscr{F}_k(S)\cap\Gamma(S_2)$$

$\mathscr{B}^1=\{P_1(B):B\in\mathscr{B}\}$ 和 $\mathscr{C}=\{C\in\mathscr{B}^1:C\bar{\in}\mathscr{A}\}$ 如图 1 所示：

显然有：1）$\mathscr{F}_k(S)=\mathscr{B}\cup\mathscr{A}$，

2）$\mathscr{B}^1=\mathscr{C}\cup\mathscr{A}$，

3）$\mathscr{A}$是S_1上的G_{3k}子集系．

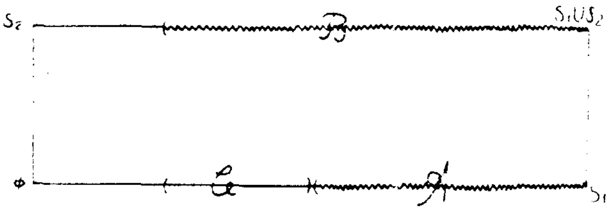

图 1

若$\mathscr{F}$为极大的G_{3k}子集系，且$\mathscr{F}\supseteq\mathscr{A}$，那么称$\mathscr{F}$为$\mathscr{A}$导出的极大的$G_{3k}$子集系．

先证：$\mathscr{A}$导出的极大的G_{3k}子集系$\mathscr{F}$的存在性．如果$\mathscr{A}$为极大的G_{3k}子集系，则$\mathscr{F}=\mathscr{A}$．否则，由条件（2），存在$A_0'\in\mathscr{B}(S_1)$，$A_0'\bar{\in}\mathscr{A}$，使得对任意的$A_1,\cdots,A_{k-1}\in\mathscr{A}$都存在某个$A_i$有$A_0'\cap A_i\neq\phi$．此时，我们得到了新的$G_{3k}$子集系$\mathscr{A}\cup\{A_0'\}$．如$\mathscr{A}\cup\{A_0'\}$是极大的$G_{3k}$子集系，则$\mathscr{F}=\mathscr{A}\cup\{A_0'\}$．若不然，由条件（2）又可以找到$A_1'\in\mathscr{B}(S_1)$使得$\mathscr{A}\cup\{A_0',A_1'\}$是$G_{3k}$子集系．如果$\mathscr{A}\cup\{A_0',A_1'\}$是极大的$G_{3k}$子集系，则取$\mathscr{F}=\mathscr{A}\cup\{A_0',A_1'\}$．否则，继续找新的$G_{3k}$子集系，由$|S_1|$为有限的，故必然有一个$G_{3k}$子集系为极大的．

其次证：$\mathscr{A}$导出的极大的G_{3k}子集系$\mathscr{F}$，必有$\mathscr{F}\subseteq\mathscr{B}'$．假设存在$A\in\mathscr{F}-\mathscr{B}'$，由$\mathscr{B}'$的定义，$A\cup S_2\bar{\in}\mathscr{F}_k(S)$，于是，存在$A_1,\cdots,A_{k-1}\in\mathscr{F}_k(S)$使得$A_1,\cdots,A_{k-1}$与$A\cup S_2$两两不相交．此时，有$A_1,\cdots,A_{k-1},A\in\mathscr{F}$，且它们是两两不相交的．与$\mathscr{F}$的定义矛盾．

最后证：存在一$\mathscr{A}$导出的极大的G_{3k}子集系$\mathscr{F}$，具有

$$|\mathscr{F}|<|\mathscr{A}|+\frac{|\mathscr{C}|}{2} \tag{2.4}$$

我们对$\mathscr{C}$的基数进行归纳．当$|\mathscr{C}|=0$时，$\mathscr{A}$就是极大的G_{3k}子集系，显然，结论成立．

当$0<|\mathscr{C}|<t$成立时，证$|\mathscr{C}|=t$的情况．令$\mathscr{A}(C_i)=\{C\in\mathscr{C}$：存在$A_1,\cdots,A_{k-2}\in\mathscr{A}$使得$A_1,\cdots,A_{k-2}$，C 与$C_i$是两两不相交的$\}$ $C_i\in\mathscr{C}$,显然有：i）若$C\in\mathscr{A}(C_i)$，则$C_i\in\mathscr{A}(C)$；

ii）令$\mathscr{F}-\mathscr{A}=\mathscr{C}'$，且$C\in\mathscr{C}'$,则$\mathscr{A}(C)\cap\mathscr{C}'=\phi$称$\mathscr{C}'$为$\mathscr{A}$的可补集；iii）$\mathscr{C}'$为$\mathscr{A}$的的可补集，则$\mathscr{C}'\subseteq\mathscr{C}$.

设$C^*\in\mathscr{C}$，使得$|\mathscr{A}(C^*)|=\min\{|\mathscr{A}(C)|:C\in\mathscr{C}\}$,并且令，$\tilde{\mathscr{A}}(C^*)=\{C\cup S_2:C\in\mathscr{A}(C^*)\}$

和$\widetilde{\mathscr{B}}_1=\mathscr{B}-\widetilde{\mathscr{A}}(C^*)$．显然，$\widetilde{\mathscr{B}}_1\cup\mathscr{A}$仍为$G_{3k}$子集系，但未必是极大的．设$\mathscr{F}_1'$为$\widetilde{\mathscr{B}}_1\cup\mathscr{A}$导出的极大的$G_{3k}$子集系．记$\mathscr{C}_1=\mathscr{F}_1'-\widetilde{\mathscr{B}}_1\cup\mathscr{A}$，于是，$\mathscr{C}_1\subseteq\mathscr{C}$．若不然，$A\in\mathscr{C}_1-\mathscr{C}$．由$\mathscr{C}$的定义$A\cup S_2\overline{\in}\mathscr{F}_k(S)$，存在$A_1,\cdots,A_{k-1}\in\mathscr{F}_k(S)$与$A\cup S_2$是两两不相交的，因为$A_1,\cdots,A_{k-1}$与$S_2$是两两不相交的，所以，$A_1,\cdots,A_{k-1}\in\mathscr{A}$，与$\mathscr{C}_1$是可补集发生冲突．

若$C\in\mathscr{C}_1$，则$\mathscr{A}(C)=\mathscr{A}(C^*)$，由$\mathscr{A}(C^*)$是基数最小的，所以，$\mathscr{A}(C)\supseteq\mathscr{A}(C^*)$．于是，假设存在$A\in\mathscr{A}(C)-\mathscr{A}(C^*)$，由$\mathscr{A}(C)$的定义，存在$A_1,\cdots,A_{k-2}\in\mathscr{A}$使得$A_1,\cdots,A_{k-2}$，$C$与$A$两两不相交．而$A\in\mathscr{A}(C^*)$，$A\cup S_2\in\widetilde{\mathscr{B}}_1$．从而得，$A_1,\cdots,A_{k-2}$，$A\cup S_2$，$C\in\mathscr{F}_1'$是两两不相交的．与$\mathscr{F}_1'$的定义矛盾．

如果$|\mathscr{C}_1|\leqslant|\mathscr{A}(C^*)|$，且令$\mathscr{A}_1^*=\mathscr{F}_1'\cap\Gamma(\phi)$，$\mathscr{B}_1^*=\mathscr{F}_1'\cap\Gamma(S_2)$，$\mathscr{B}_1^{*\prime}=\{P_1(B):B\in\mathscr{B}_1^*\}$和$\mathscr{C}_1^*=\{C\in\mathscr{B}_1^{*\prime}:C\overline{\in}\mathscr{A}_1^*\}$．又$|\mathscr{C}_1^*|<t$．由归纳假定，存在$\mathscr{A}_1^*$的导出极大的$G_{3k}$子集系$\mathscr{F}_1^{1*}$，具有$|\mathscr{F}_1|\leqslant|\mathscr{A}_1^*|+\frac{|\mathscr{C}_1^*|}{2}$．$\mathscr{F}_1$也是$\mathscr{A}$导出的极大$G_{3k}$子集系，由于$|\mathscr{A}_1^*|=|\mathscr{A}|+|\mathscr{C}_1|$和$|\mathscr{C}|=|\mathscr{A}(C^*)|+|\mathscr{C}_1|+|\mathscr{C}_1^*|$，所以，$|\mathscr{F}_1|\leqslant|\mathscr{A}|+\frac{|\mathscr{C}|}{2}$．如图 2 所示．

如果$|\mathscr{C}_1|>|\mathscr{A}(C^*)|$，令$\widetilde{\mathscr{C}}_1=\{C\cup S_2:C\in\mathscr{C}_1\}$和$\widetilde{\mathscr{B}}_2=\mathscr{B}-\widetilde{\mathscr{C}}_1$，显然，$\widetilde{\mathscr{B}}_2\cup\mathscr{A}$仍为$G_{3k}$子集系，但未必是极大的，设$\mathscr{F}_2'$为$\widetilde{\mathscr{B}}_2\cup\mathscr{A}$导出的极大的$G_{3k}$子集系，可补集$\mathscr{C}_2=\mathscr{F}_2'-(\widetilde{\mathscr{B}}_2\cup\mathscr{A})\subseteq\mathscr{C}$，且$\mathscr{C}_2\subseteq\mathscr{A}(C^*)$．第一个包含式子在前面证过．下面证第二个包含式子，假设$A\in\mathscr{C}_2-\mathscr{A}(C^*)$，由$C\in\mathscr{C}_2$，蕴涵$\mathscr{A}(C)\subseteq\mathscr{C}_1$，取$C'\in\mathscr{A}(A)$，有$A\in\mathscr{A}(C')$．所以$\mathscr{A}(C')\neq\mathscr{A}(C^*)$．这与前面证过的结论："$C\in\mathscr{C}_1$则$\mathscr{A}(C)=\mathscr{A}(C^*)$"发生矛盾．

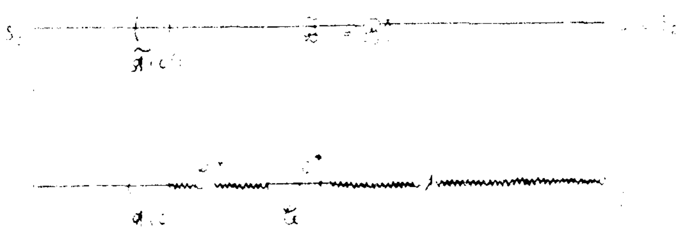

图 2

令$\mathscr{B}_2^*=\mathscr{F}_2'\cap\Gamma(S_2)$，$\mathscr{B}_2^{*\prime}=\{P_1(B):B\in\mathscr{B}_2^*\}$，

$\mathscr{A}_2^*=\mathscr{F}_2'\cap\Gamma(\phi)$，　$\mathscr{C}_2^*=\{C\in\mathscr{B}_2^{*\prime}:C\overline{\in}\mathscr{A}_2^*\}$，

且$|\mathscr{C}_2^*|<T$．由归纳假定，存在$\mathscr{A}_2^*$的导出极大的G_{3k}子集系$\mathscr{F}_2$，具有$|\mathscr{F}_2|\leqslant|\mathscr{A}_2^*|+\frac{|\mathscr{C}_2^*|}{2}$．显见，$\mathscr{F}_2$也是$\mathscr{A}$导出的极大$G_{3k}$子集系．由$|\mathscr{A}_2^*|=|\mathscr{A}|+|\mathscr{C}_2|$和$|\mathscr{C}|=|\mathscr{C}_1|+|\mathscr{C}_2|+|\mathscr{C}_2^*|$，有$|\mathscr{F}_2|\leqslant|\mathscr{A}|+\frac{|\mathscr{C}|}{2}$．这就证明了最后的结论．

由最后结论及归纳假定有

$$|\mathscr{F}_k(S)|=|\mathscr{A}|+|\mathscr{B}|\geqslant2|\mathscr{A}|+|\mathscr{C}|\geqslant2(|\mathscr{A}|+\frac{|\mathscr{C}|}{2})\geqslant2|\mathscr{F}|\geqslant2^n-2^{n-k+1},$$

我们获得了定理 2 的证明．

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参 考 文 献

〔1〕D. Kleitman, Maximal number of subsets of a finite set no K of which are pairwise disjoint, J. Combin. Theory, 5(1968), 152.

〔2〕P. Erdos and D. Kleitman, Extremal problems among subsets of a set, Discrete Math., 8(1974), 281.

The Smallest Size of a Maximal Family of Subsets of a Finite Set No k of Which are Pairwise Disjoint

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Abstract

Let S be a finite set with n elements, and $\mathscr{F}_k(s)$ be a maximal family of subsets of S no k of which are pairwise disjoint. i.e for any $A_1, \cdots, A_k \in \mathscr{F}_k(s)$, there exist A_i and $A_j (i \neq j)$ such that $A_i \cap A_j \neq \phi$; and for $A_0 \subseteq S$, $A_0 \overline{\in} \mathscr{F}_k(S)$, there exist $A_1', \cdots, A_{k-1}'$ in $\mathscr{F}_k(s)$ such that $A_1', \cdots, A_{k-1}'$ and A_0 are pairwise disjoint. How large can $\mathscr{F}_k(s)$ be? An unper bound on size of $\mathscr{F}_k(S)$ has been obtained by kleitman (see [1]), but lower bound remains open. P.Erdos and D.Kleit-man asked if the smallest size of a maximal family of subsets of S no k of which are pairwise disjoint is equal to $2^n - 2^{n-k}$ (see [2']) This paper answers the question nogatively, and the smallest size of $\mathscr{F}_k(s)$ is determined. The main results can be described as follows.

Theorem I $\min|\mathscr{F}_k(s)| < 2^n - 2^{n-k+1}$, where the minimum is taken over all maximal family of subsets of S no k of which are pairwise disjoint.

Theorem 2 $\min|\mathscr{F}_k(s)| = 2^n - 2^{n-k+1}$.