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Pansystems Whole-Part Relation Analysis and Triple Covering Systems*

Zhu Xuding (朱绪鼎)
(Wuhan Digital Engineering Institute)

Abstract Let $A_n=\{1,2,\cdots,n\}$ and let $\mathscr{B}=\{B_1, B_2, \cdots, B_r\}$ where B_1、B_2, $\cdots$, B_r are subsets of A_n each of size m. $\mathscr{B}$ is said to cover all the triples (i, j, k), $1\leqslant i<j<k\leqslant n$, if for each triple set $\{i, j, k\}$ there exists a t, $1\leqslant t\leqslant r$, such that $\{i, j, k\}\subseteq B_t$. Denote by $V(m, n)$ the minimum possible cardinality of such a system $\mathscr{B}$. It is shown that if $\frac{m}{n}>\frac{2}{3}$, then $V(m, n)$ is a function of the fraction $\frac{m}{n}$ only and the values of $V(m, n)$ are determined for all m, n with $\frac{m}{n}\geqslant\frac{2}{3}$. The value of $V(m, n)$ for $\frac{m}{n}<\frac{2}{3}$ is also discussed.

I. Introduction

It is not unusual that one needs to analyze great number of data. One of the useful methods to do so is to divide these data into a number of classes and this process is called partition. Pansystems methodology, dealing with large scale system, studied the so called partitions thoroughly and introduced the concepts semi-equivalence partition and panpartition to cope with undistinct partitions, that is two different classes may possibly intersect each other. Both semi-equivalence partition and panpartition investigate pansystem whole-part relations and properties of certain collections of subsets of a given universe. In much the same way, pair covering systems[1] discuss a special kind of collections of subsets of a given universe.

An (m, n) pair-covering system is a collection of subsets of A_n, say $\mathscr{B}=\{B_1, B_2, \cdots, B_r\}$, each of size m and for any pair set $\{i, j\}\subseteq A_n$ there exists a $t(1\leqslant t\leqslant r)$ such that $\{i, j\}\subseteq B_t$, where $A_n=\{1,2,\cdots,n\}$. Denote by $N(m, n)$ the minimum possible cardinality of an (m, n) pair-covering system $\mathscr{B}$, M. K. Fort Jr. and G. A. Hedlund determined the values of $N(3, n)$ exactly for all n[5]. Their results have been further extended to a more general function by H. Hanani[6]. Y. Shiloach, U. Vishkin and S. Zaks determined the values of $N(m, n)$ for all m, n with $\frac{m}{n}\geqslant\frac{1}{2}$. We present their result here as it is needed in the proof of theorems of this paper.

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Theorem 1[1] If $\frac{m}{n}\geqslant\frac{1}{2}$, then $N(m, n)$ is a function of $\frac{m}{n}$ only and

$$N(m, n)=\begin{cases} 3 & \text{if } \frac{2}{3}\leqslant\frac{m}{n}<1, \\ 4 & \text{if } \frac{3}{5}\leqslant\frac{m}{n}<\frac{2}{3}, \\ 5 & \text{if } \frac{5}{9}\leqslant\frac{m}{n}<\frac{3}{5}, \\ 6 & \text{if } \frac{1}{2}\leqslant\frac{m}{n}<\frac{5}{9}. \end{cases}$$

An (m, n)-pair covering system can also be defined as a collection of subsets of A_n of size m, say $\mathscr{B}=\{B_1, B_2, \cdots, B_r\}$, satisfying: (a) $\bigcup_{i=1}^{r} B_i = A_n$; (b) $(\bigcup_{i=1}^{r} B_i)^2=\bigcup_{i=1}^{r} B_i^2$.

From this definition, we can see that pair covering systems deal with a certain pansymmetry, that is, the union of the embodiments of a class of subsets is equal to the embodiment of the union of these subsets. By extending the concepts partition, panpartition and pair covering system, this papes is devoted to triple covering systems.

Definition 1 An (m, n)-triple covering system $\mathscr{B}$ ((m, n)-$t.c.s.$) is a collection of subsets of A_n, say $\mathscr{B}=\{B_1, B_2, \cdots, B_n\}$, each of size m and for each triple set $\{i, j, k\}\subseteq A_n$ there exists a $t(1\leqslant t\leqslant r)$ such that $\{i, j, k\}\subseteq B_t$, where $A_n=\{1, 2, \cdots, n\}$.

$\mathscr{B}$ is a minimum (m, n)-t.c.s. if there is no other (m, n)-t.c.s. with smaller cardinality. The cardinality of a minimum (m, n)-t.c.s. is a function of m and n only and is denoted by $V(m, n)$.

The value of $V(m, n)$ has also been investigated by many authors. W.H. Mills determined the value of $V(4, n)$ for $n\not\equiv 7(\bmod 12)$, and R. G. Stanton proved that there is an obvious lower bound $L(4, n)$ such that for infinite number of $n\equiv 7(\bmod 12)$, $V(4, n)\leqslant L(4, n)+1$. This paper discusses the value of $V(m, n)$ from another viewpoint and following results are obtained.

2. Main Results

Theorem 2 If $\frac{m}{n}>\frac{2}{3}$, then $V(m, n)$ is a function of the fraction $\frac{m}{n}$ only and

$$V(m, n)=\begin{cases} 4 & \text{if } \frac{3}{4}\leqslant\frac{m}{n}<1, \\ 5 & \text{if } \frac{5}{7}\leqslant\frac{m}{n}<\frac{3}{4}, \\ 6 & \text{if } \frac{2}{3}<\frac{m}{n}<\frac{5}{7}. \end{cases}$$

Theorem 3 If $\frac{m}{n}=\frac{2}{3}$, then

$$V(m, n) = \begin{cases} 6 & \text{if } m = 4k, \ n = 6k \\ 7 & \text{if } m = 4k+2, \ n = 6k+3 \end{cases}$$

where k is any positive integer.

This theorem also asserts that $V(m, n)$ is not a function of the fraction $\frac{m}{n}$ only if $\frac{m}{n} = \frac{2}{3}$.

3. Proof of Theorem 2

3.1 On Lower Bound

Lemma 1 a) $V(m, n) \geqslant 4$, if $\frac{m}{n} < 1$,

b) $V(m, n) \geqslant 5$, if $\frac{m}{n} < \frac{3}{4}$,

c) $V(m, n) \geqslant 6$, if $\frac{m}{n} < \frac{5}{7}$,

d) $V(m, n) \geqslant 7$, if $\frac{m}{n} < \frac{2}{3}$.

Proof a) If $V(m, n) < 4$, Let $\mathscr{B} = \{B_1, B_2, B_3\}$ be an (m, n)-t.c.s. where $\frac{m}{n} < 1$. Obviously, at least two of $B_i's$ are different, say $B_1 \neq B_2$. Let $x_1 \in B_1 - B_2$, $x_2 \in B_2 - B_1$, $x_3 \in A_n - B_3$, the triple set $\{x_1, x_2, x_3\}$ is not covered by any of $B_i's$.

b) Otherwise, there is an (m, n)-t.c.s. $\mathscr{B} = \{B_1, B_2, B_3, B_4\}$, where $\frac{m}{n} < \frac{3}{4}$. Then the average number of occurrences of an element in the $B_i's$ is $4m/n < 3$. Thus there is at least one element belongs to exactly two of $B_i's$, say $1 \in B_1 \cap B_2$. According to the definition, this implies that $\overline{\mathscr{B}} = \{B_1 - \{1\}, B_2 - \{1\}\}$ is an $(m-1, n-1)$-pair-covering system, which contradicts to theorem 1.

In addition, we proved that if $\mathscr{B} = \{B_1, B_2, \cdots, B_r\}$ is an (m, n)-t.c.s., then every element of A_n belongs to at least three of $B_i's$.

c) Otherwise, there is an (m, n)-t.c.s. $\mathscr{B} = \{B_1, B_2, \cdots, B_5\}$ for $\frac{m}{n} < \frac{5}{7}$. The average number of occurrences of an element in the $B_i's$ is $\frac{5m}{n} < \frac{25}{7} < 4$. Thus there is at least one element belongs to exactly three of the $B_i's$. Assume this element is x and it occurs in B_1, B_2, B_3. Then equations

$$B_i \cup B_j = A_n \qquad i \neq j, \ i, j = 1, 2, 3$$

must hold.

Otherwise, suppose $B_1 \cup B_2 \neq A_n$. Let

$a \in A_n - B_1 \cup B_2$, $b \in A_n - B_3$, it is quite obvious that the triple set $\{x, a, b\}$ is covered by none of the $B_i's$ $(i = 1, 2, \cdots, 5)$. A contradiction.

Let $C_1 = B_1 - B_2$, $C_2 = B_2 - B_1$, then $|C_1| = |C_2| > \frac{2}{7}n$ and $B_3 \supseteq C_1 \cup C_2$. Thus

$$|B_1 \cap B_2 - B_3| = |A_n - (C_1 \cup C_2) - B_3| = |A_n - B_3| > \frac{2}{7}n,$$

$$|C_1 \cup C_2 \cup (B_1 \cap B_2 - B_3)| = |C_1| + |C_2| + |B_1 \cap B_2 - B_3| > \frac{6}{7}n.$$

Clearly, any element of $C_1 \times C_2 \times (B_1 \cap B_2 - B_3)$ is covered by none of B_1, B_2, B_3.

Since $|B_4| = |B_5| = m < \frac{5}{7}n$, let $u \in C_1 \cup C_2 \cup (B_1 \cap B_2 - B_3) - B_4$, $v \in C_1 \cup C_2 \cup (B_1 \cap B_2 - B_3) - B_5$. Without losing generality, suppose $u \in C_1$, then it is necessary that

$u \in B_5$ and $B_5 \supseteq C_2 \cup (B_1 \cap B_2 - B_3)$. This implies $|B_5 \cap C_1| < \frac{1}{7}n$. If $v \in C_1$, with the same discussion, we have $|B_4 \cap C_1| < \frac{1}{7}n$, which contradicts the fact that $|(B_4 \cup B_5) \cap C_1| = |C_1| > \frac{2}{7}n$. If $v \notin C_1$, suppose $v \in C_2$, let w be an element of $B_1 \cap B_2 - B_3$, it is obvious that $\{u, v, w\}$ is covered by none of the B's $(i = 1, 2, \cdots, 5)$.

d) Otherwise, let $\mathscr{B} = \{B_1, B_2, \cdots, B_6\}$ be an (m, n) - t.c.s. for $\frac{m}{n} < \frac{2}{3}$. The average number of occurrences of an element in the $B_i's$ is $6m/n < 4$. Thus there is at least one element occurs exactly in three of the $B_i's$. Let these three $B_i's$ are B_1, B_2, B_3, with the same discussion in c), the equations

$$B_i \cup B_j = A_n \qquad i \neq j, \ i, j = 1, 2, 3$$

must hold.

But the total number of occurrences of elements in B_1, B_2, B_3 is $3m < 2n$. Thus there is at least one element occurs in exactly one of B_1, B_2, B_3. Suppose this element occurs in B_1, then $B_2 \cup B_3 \neq A_n$. A contradiction.

3.2 On Upper Bound

Lemma 2 $V(m, n) \geq V(km, kn)$ for any positive integer k.

Proof Suppose $\mathscr{B} = \{B_1, B_2, \cdots, B_r\}$ is an (m, n) - t.c.s. and let $C_1, C_2, \cdots, C_n$ be a partition of A_{kn} such that $|C_i| = k$ for $i = 1, 2, \cdots, n$. It is obvious that $\overline{\mathscr{B}} = \{\overline{B_1}, \overline{B_2}, \cdots, \overline{B_r}\}$ is a (km, kn) - t. c. s. where

$$\overline{B_i} = \bigcup_{j \in B_i} C_j \qquad (i = 1, 2, \cdots, r).$$

Lemma 3 $V(m, n) \leq 4$ if $\frac{m}{n} \geq \frac{3}{4}$.

Proof It is obvious that $V(3, 4) = \binom{4}{3} = 4$ and it follows from lemma 2 that $V(3k, 4k) \leq 4$. If $m = 3k + 1$ $(m = 3k + 2)$, then $n \leq [\frac{4}{3}m] = 4k + 1$ $(n \leq 4k + 2)$. Suppose $\mathscr{B} = \{B_1, B_2, B_3, B_4\}$ is a $(3k, 4k)$ - t.c.s., it is easy to prove that $\mathscr{B}' = \{B_1', B_2', B_3', B_4'\}$, $\mathscr{B}'' = \{B_1'', B_2'', B_3'', B_4''\}$ are $(3k + 1, 4k + 1)$ - t.c.s. and $(3k + 2, 4k + 2)$ - t.c.s. respectively, where $B_i' = B_i \cup \{4k + 1\}$, $B_i'' = B_i' \cup \{4k + 2\}$ $(i = 1, 2, 3, 4)$.

In fact, we proved that for any positive integer t, $V(m + t, n + t) \leq V(m, n$.

Lemma 4 $V(m, n) \leq 5$ if $\frac{m}{n} \geq \frac{5}{7}$. **Proof** Let $B_1 = \{1, 2, 3, 4, 5\}$, $B_2 = \{3, 4, 5, 6, 7\}$, $B_3 = \{1, 2, 3, 6, 7\}$, $B_4 = \{1, 2, 4, 6, 7\}$, $B_5 = \{1, 2, 5, 6, 7\}$. It is easy to prove that $\mathscr{B} = \{B_1, B_2, \cdots, B_5\}$ is a (5, 7) - t. c. s. and it follows from lemma 2 that $V(5k, 7k) \leq 5$.

If $m = 5k + 1$, then $n \leq [\frac{7}{5}m] = 7k + 1$, if $m = 5k + 2$, then $n \leq [\frac{7}{5}m] = 7k + 2$. As we have shown $V(m + t, n + t) \leq V(m, n)$, it follows that $V(5k + 1, 7k + 1)$, $V(5k + 2, 7k + 2) \leq 5$.

If $m = 5k + 3$, then $n \leq [\frac{7}{5}m] = 7k + 4$. Let $B_1 = \{1, 2, \cdots, m\}$, $B_2 = \{n - m + 1, \cdots, n\}$, $B_3 = \{1, 2, \cdots, n - m, m + 1, \cdots, n, n - m + 1, \cdots, n - m + k + 1\}$, $B_4 = \{1, \cdots, n - m, m + 1, \cdots, n, n - m + k + 2, \cdots, n - m + 2k + 2\}$, $B_5 = \{1, \cdots, n - m, m + 1, \cdots, n, n - m + k + 2,$

$\cdots, m\}$, it is easy to check that $\mathscr{B}=\{B_1, B_2, \cdots, B_5\}$ is a $(5k+3, 7k+4)$ - t.c.s. where $n=7k+4$.

If $m=5k+4$, then $n\le[\frac{7}{5}m]=7k+5$ and it follows that $V(5k+4, 7k+5)\le V(5k+3, 7k+4)\le 5$.

Lemma 5 $V(m, n)\le 6$ if $\frac{m}{n}\ge\frac{2}{3}$ and $4\nmid m-2$, $V(m, n)\le 6$ if $\frac{m}{n}>\frac{2}{3}$.

Proof Let $B_1=\{1,2,3,4\}$, $B_2=\{1,2,5,6\}$, $B_3=\{1,3,5,6\}$, $B_4=\{1,4,5,6\}$, $B_5=\{2,3,4,5\}$, $B_6=\{2,3,4,6\}$, Only twenty cases should be checked to show that $\mathscr{B}=\{B_1, B_2, \cdots, B_6\}$ is a $(4, 6)$ - t. c. s. and it follows that $V(4k, 6k)\le 6$.

If $m=4k+1$, then $n\le[\frac{3}{2}m]=6k+1$, it follows that $V(4k+1, 6k+1)\le V(4k, 6k)\le 6$.

If $m=4k+2$, then $4\mid m-2$, $n<\frac{3}{2}m=6k+3$. Thus $n\le 6k+2$, it follows that $V(4k+2, 6k+2)\le V(4k, 6k)\le 6$.

If $m=4k+3$, then $n\le[\frac{3}{2}m]=6k+4$. Let $B_1=\{1, \cdots, 4k+2, 6k+4\}$, $B_2=\{1, \cdots, k, 2k+2, \cdots, 3k+2, 4k+3, \cdots, 6k+4\}$, $B_3=\{1, \cdots, 2k+1, 4k+3, \cdots, 6k+4\}$, $B_4=\{1, \cdots, k+1, 3k+3, \cdots, 4k+2, 4k+3, \cdots, 6k+4\}$, $B_5=\{k+1, \cdots, 5k+2, 6k+4\}$, $B_6=\{k+1, \cdots, 4k+2, 5k+3, \cdots, 6k+3\}$. It can be proved that $\mathscr{B}=\{B_1, B_2, \cdots, B_6\}$ is a $(4k+3, 6k+4)$ - t. c. s.

4. Proof of Theorem 3

Lemma 5 completed the proof of the first part of theorem 3. Now we only need to prove $V(4k+2, 6k+3)=7$ for all positive integers k.

Lemma 6 $V(4k+2, 6k+3)\ge 7$, where k is any positive integer.

Proof Otherwise, let $\mathscr{B}=\{B_1, B_2, \cdots, B_6\}$ be a $(4k+2, 6k+3)$ - t.c.s. Thus the average number of occurence of an element in the $B_i's$ is

$$\frac{6(4k+2)}{6k+3}=4.$$

If there is an element belongs to only three of the $B_i's$, it may lead to a contradiction (see the proof of lemma 1, d)). This implies that every element belongs to exactly four of the $B_i's$.

Let $\mathscr{C}=\{C \mid C\subset\{1,2,3,4,5,6\}, |C|=4\}$ and let T: $\mathscr{C}\to P(A_n)$ be defined as

$$T(C)=\bigcap_{i\in C}B_i,$$

where $P(A_n)$ is the power set of A_n.

Let us keep in mind that $B_i=\bigcup T(C)(T(C)\subset B_i)$, $i\in C$ is equivalent to $T(C)\subset B_i$.

Since every element of A_n belongs to exactly four of the $B_i's$, the following equations hold:

$$T(C_1)\cap T(C_2)=\varnothing \text{ if } C_1\ne C_2, \bigcup_{C\in\mathscr{C}}T(C)=A_n$$

Lemma 7 $|T(\mathrm{C})| \leqslant k, \ \forall \mathrm{C} \in \mathscr{C}$.

Proof Suppose $|T(\mathrm{C})| > 0$ and $\mathrm{C} = \{i_1, i_2, i_3, i_4\}$. Let $\{x, y\} \subseteq \mathrm{A}_n - T(\mathrm{C})$, if there exists no t $(1 \leqslant t \leqslant 4)$, such that $\{x, y\} \subseteq \mathrm{B}_{i_t}$, then it is quite obvious that $\{u, x, y\} \not\subseteq \mathrm{B}_i$ for any $i \in \{1, 2, \cdots, 6\}$, $u \in T(\mathrm{C})$. This contradicts to the assumption that $\{\mathrm{B}_1, \mathrm{B}_2, \cdots, \mathrm{B}_6\}$ is a (m, n)-t.c.s. Thus every pair set $\{x, y\} \subseteq \mathrm{B}_{i_t}$ for a certain $t(1 \leqslant t \leqslant 4)$, where $\{x, y\} \subseteq \mathrm{A}_n - T(\mathrm{C})$. This implies $\mathscr{B}' = \{\mathrm{B}'_1, \mathrm{B}'_2, \mathrm{B}'_3, \mathrm{B}'_4\}$ is a $(m - |T(\mathrm{C})|, n - |T(\mathrm{C})|)$-pair covering system, where $\mathrm{B}'_t = \mathrm{B}_{i_t} - T(\mathrm{C})$ $(t = 1, 2, 3, 4)$. According to theorem 1,

$$\frac{m - |T(\mathrm{C})|}{n - |T(\mathrm{C})|} \geqslant \frac{3}{5},$$

which implies $5(4k + 2 - |T(\mathrm{C})|) \geqslant 3(6k + 3 - |T(\mathrm{C})|)$, $|T(\mathrm{C})| \leqslant k + \frac{1}{2}$. Thus $|T(\mathrm{C})| \leqslant k$.

Let $\mathscr{P} = \{\mathrm{C} \mid \mathrm{C} \in \mathscr{C}, \ T(\mathrm{C}) \neq \varnothing\}$, it follows from the fact $\left|\bigcup_{\mathrm{C} \in \mathscr{C}} T(\mathrm{C})\right| = |\mathrm{A}_n| = 6k + 3$, $|T(\mathrm{C})| \leqslant k$, $\forall \mathrm{C} \in \mathscr{C}$ that $|\mathscr{P}| \geqslant 7$.

Lemma 8 If $\mathrm{C}_i, \mathrm{C}_j, \mathrm{C}_k \in \mathscr{P}$, then $\mathrm{C}_i \cap \mathrm{C}_j \cap \mathrm{C}_k \neq \varnothing$.

Proof Let $x \in T(\mathrm{C}_1)$, $y \in T(\mathrm{C}_2)$, $z \in T(\mathrm{C}_3)$. Since there exists a $\mathrm{B}_t, (1 \leqslant t \leqslant 6)$, such that $\{x, y, z\} \subseteq \mathrm{B}_t$, which implies $t \in \mathrm{C}_1 \cap \mathrm{C}_2 \cap \mathrm{C}_3$. Thus $\mathrm{C}_i \cap \mathrm{C}_j \cap \mathrm{C}_k \neq \varnothing$.

Lemma 9 $|\mathscr{P}| \leqslant 10$.

Proof Otherwise, let $\{\mathrm{C}_1, \mathrm{C}_2, \cdots, \mathrm{C}_{11}\} \subseteq \mathscr{P}$. If is easy to see that for any $\mathrm{C} \in \mathscr{C}$, there are three pairs, say C'_i, $\mathrm{C}''_i \in \mathscr{C}$ $(i = 1, 2, 3)$, such that $\mathrm{C} \cap \mathrm{C}'_i \cap \mathrm{C}''_i = \varnothing$ and for any pair $\mathrm{C}_i, \mathrm{C}_j \in \mathscr{C}$, there is at most one element $\mathrm{C}_k \in \mathscr{C}$ makes $\mathrm{C}_i \cap \mathrm{C}_j \cap \mathrm{C}_k = \varnothing$. This implies there are thirty-three pairs, say $\mathrm{C}'_k, \mathrm{C}''_k \in \mathscr{C}$ $(k = 1, 2, \cdots, 33)$, such that for any given k $(1 \leqslant k \leqslant 33)$ $\mathrm{C}'_k \cap \mathrm{C}''_k \cap \mathrm{C}_i = \varnothing$ for a certain i $(1 \leqslant i \leqslant 11)$.

According to lemma 8, for any k $(1 \leqslant k \leqslant 33)$, at least one of C'_k, C''_k belongs to $\mathscr{C} - \mathscr{P}$.

For a given pair C', $\mathrm{C}'' \in \mathscr{C}$, C', C'' is said to possess Property A if

$$\exists \, \mathrm{C} \in \mathscr{C}, \text{ such that } \mathrm{C} \cap \mathrm{C}' \cap \mathrm{C}'' = \varnothing \qquad (\mathrm{A})$$

The discussion above shows that there are thirty-three pairs, C'_k, C''_k $(k = 1, 2, \cdots, 33)$, possess Property A and for each k $(1 \leqslant k \leqslant 33)$, at least one of C'_k, C''_k belongs to $\mathscr{C} - \mathscr{P}$.

But since $|\mathscr{C}| = \binom{6}{4} = 15$, $|\mathscr{C} - \mathscr{P}| = |\mathscr{C}| - |\mathscr{P}| \leqslant 4$ and for any $\mathrm{C}' \in \mathscr{C}$, there exist exactly six $\mathrm{C}''_j \in \mathscr{C}$ $(j = 1, 2, \cdots, 6)$ such that C', C''_j possesses Property A, the total number of pairs of $\mathscr{C}$, any one of which contains at least one element of $\mathscr{C} - \mathscr{P}$ and possesses Property A, is at most equal to $4 \times 6 = 24$.

This leads to a contradiction and lemma 9 is proved.

Lemma10 Each element of $\{1,2,\cdots,6\}$ belongs to at most $|\mathscr{P}|-3$ elements of $\mathscr{P}$.

Proof Otherwise, there is an element, say i $(1<i<6)$, belongs to $|\mathscr{P}|-2$ elements of $\mathscr{P}$. Let the two elements which does not contain i be C_1, C_2, then

$$T(C)\subseteq B_i \text{ for all } C\in\mathscr{P}-\{C_1, C_2\}.$$

Thus

$$A_n - T(C_1)\cup T(C_2 = \bigcup_{C\in\mathscr{C}} T(C) - T(C_1)\cup T(C_2) = \bigcup_{C\in\mathscr{P}} T(C) - T(C_1)\cup T(C_2) = B_i$$

This implies

$$|A_n - T(C_1)\cup T(C_2)| = |A_n| - |T(C_1)\cup T(C_2)| = |B_i| = 4k+2.$$

$$|T(C_1)\cup T(C_2)| = |T(C_1)| + |T(C_2)| = 2k+1.$$

This contradicts to lemma 7.

Lemma 11 $|\mathscr{P}|\neq 7$, $|\mathscr{P}|\neq 8$.

Proof If $|\mathscr{P}|=7$, then it follows from lemma 10 that each element of $\{1, 2,\cdots,6\}$ belongs to at most four elements of $\mathscr{P}$. Thus the total number of occurrences of $\{1,2,\cdots,6\}$ in elements of $\mathscr{P}$ is at most equal to $4\times 6=24$, but this number should equal to $4\times 7=28$. This is a contradiction. Thus $|\mathscr{P}|\neq 7$.

$|\mathscr{P}|\neq 8$ is proved in much the same way.

To complete the proof of Lemma 6, there are only two cases need to be ruled out.

Case 1. $|\mathscr{P}|=9$. Let $\mathscr{P}=\{C_1, C_2, \cdots, C_9\}$.

With much the same discussion as that of lemma 11, it can be proved that every element of $\{1,2,\cdots,6\}$ occurs in exactly six elements of $\mathscr{P}$. But the following lemma asserts that this is impossible.

Lemma 12 There are not 9 different subsets of $\{1,2,\cdots,6\}$, say C_1, C_2, $\cdots$, C_9, satisfying:

(1) $|C_i|=4$ for $i=1,2,\cdots,9$; (2) each element of $\{1,2,\cdots,6\}$ occurs in exactly six of the $C_i's$; (3) $C_i\cap C_j\cap C_k\neq\varnothing$ $1<i<j<k<9$.

Proof If there exist such 9 subsets, then $|C_i\cap C_j\cap C_k\cap C_l|<2$, $1<i<j<k<l<9$. Otherwise, without a loss of generality, let $\{1, 2\}\subseteq C_1\cap C_2\cap C_3\cap C_4$. Then at least one of C_5, C_6, C_7, C_8, C_9, let us say C_i, contains neither 1 nor 2. (Because each of 1,2 belongs to two of C_5, C_6, C_7, C_8, C_9), Thus $C_i=\{3,4,5,6\}$.

Clearly, $C_1-\{1,2\}$, $C_2-\{1,2\}$, $C_3-\{1,2\}$, $C_4-\{1,2\}$ are four different subsets of $\{3,4,5,6\}$ each of size 2. It is easy to see that there are 6 subsets of $\{3,4,5,6\}$ with size 2 and these six subsets are three disjoint pairs, so every four subsets of $\{3,4,5,6\}$ with size 2 contains at least one of the three pairs. Suppose $(C_k-\{1,2\})\cap(C_j-\{1,2\})=\varnothing$, $1<k<j<4$, it follows that $C_k\cap C_j\cap C_i=\varnothing$. suppose $1\in C_1\cap\cdots\cap C_6$, the average occurrence of 2,3,4,5,6 in $C_1,\cdots,C_6$ is $\frac{18}{5}>3$.

Then at Least one of them occurs in four of $C_1, \cdots, C_6$. *A* contradiction.

Case 2. $|\mathscr{P}| = 10$. Let $\mathscr{P} = \{C_1, C_2, \cdots, C_{10}\}$.

With much the same discussion as that of lemma 11, 12, we can show that:

(1) each element of $\{1, 2, \cdots, 6\}$ belongs to at most seven of the C_i's and at least 4 elements of $\{1, 2, \cdots, 6\}$ occurs in seven of the C_i's.

(2) the cardinality of the intersection of any five C_i's is less than 2, *i.e.* $|C_i \cap C_j \cap C_k \cap C_l \cap C_h| < 2$ for any $1 \leqslant i < j < k < l < h \leqslant 10$.

Without a loss of generality, let each of $1, 2, 3, 4$ occurs in 7 of the C_i's, and let $1 \in C_1 \cap C_2 \cap \cdots \cap C_7$. Then each of $2, 3, 4$ occurs in at most four of C_1, C_2, $\cdots$, C_7, (see (2)). Since $2, 3, 4$ occurs in seven of the C_i's, it follows that $\{2, 3, 4\} \subseteq C_8 \cap C_9 \cap C_{11}$ and each of $2, 3, 4$ occurs in exactly four of $C_1, C_2, \cdots, C_7$. Let $2 \in C_{i_1} \cap C_{i_2} \cap C_{i_3} \cap C_{i_4}$, $3 \in C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}$, $4 \in C_{k_1} \cap C_{k_2} \cap C_{k_3} \cap C_{k_4}$, where $i_1, \cdots, i_4, j_1, \cdots, j_4, k_1, \cdots, k_4 \in \{1, 2, \cdots, 7\}$.

If $\{i_1, \cdots, i_4\} \cup \{j_1, \cdots, j_4\} \neq \{1, 2, \cdots, 7\}$, then $|\{i_1, \cdots, i_4\} \cap \{j_1, \cdots, j_4\}| \geqslant 2$. Suppose $\{u, v\} \subseteq \{i_1, \cdots, i_4\} \cap \{j_1, \cdots, j_4\}$, $1 \leqslant u < v \leqslant 7$, it follows that $\{2, 3\} \subseteq C_u \cap C_v \cap C_8 \cap C_9 \cap C_{10}$. This contradicts to (2).

If $\{i_1, \cdots, i_4\} \cup \{j_1, \cdots, j_4\} = \{1, 2, \cdots, 7\}$, then either $|\{i_1, \cdots, i_4\} \cap \{k_1, \cdots, k_4\}| \geqslant 2$, or $|\{j_1, \cdots, j_4\} \cap \{k_1, \cdots, k_4\}| \geqslant 2$. This also leads to a contradiction.

The proof of lemma 6 is completed.

Lemma 13 $V(4k+2, 6k+3) = 7$.

Proof Let $B_1 = \{1, \cdots, 4k+2\}$, $B_2 = \{1, \cdots, k, 2k+2, \cdots, 3k+2, 4k+3, \cdots, 6k+3\}$, $B_3 = \{1, \cdots, 2k+1, 4k+3, \cdots, 6k+3\}$, $B_4 = \{1, \cdots, k+1, 3k+3, \cdots, 6k+3\}$, $B_5 = \{k+1, \cdots, 5k+2\}$, $B_6 = \{k+1, \cdots, 4k+1, 5k+3, \cdots, 6k+3\}$, $B_7 = \{k+2, \cdots, 4k+2, 5k+3, \cdots, 6k+3\}$. Then $\mathscr{B} = \{B_1, B_2, \cdots, B_7\}$ is a $(4k+2, 6k+3)$-t.c.s.

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