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On Some Distributional Multiplicative Products*

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Summary

We extend the unidimensional distributional multiplicative products due to B. Fisher (of. [1], [2], [3] and [4]) to certain kinds of n-dimensional distributions called "anticausal" and "causal" distributions.

We evaluate some multiplicative products, such as $P^r_+ \cdot \delta^{(r)}(P)$, $P_+^{-r-\frac{1}{2}} \cdot P_-^{-r-\frac{1}{2}}$, $\{(\operatorname{sgn} P)\,|P|^\lambda\} \cdot P^{2r+1}$, where r and λ verify certain conditions (of. formulae (I,3;2), (I,3;4) and (I,3;10), respectively).

I. Introduction

I. 1. Brian Fisher proves the validity of the following formulae (of. [1], formula (1), *P.* 296, formula (2), *P.* 297 and 298, respectively)

$$x_+^\lambda \cdot x_-^{-1-\lambda} = -\frac{1}{2}\pi \operatorname{cosec} \pi\lambda\delta, \tag{I,1;1}$$

when $\lambda \neq 0, \pm 1, \pm 2, \cdots$

$$x^r_+ \cdot \delta^{(r)} = \frac{1}{2}(-1)^r\, r!\,\delta, \tag{I,1;2}$$

when $r = 0, 1, 2, \cdots$

$$x^r_+ \cdot \delta^{(r)} = \frac{1}{2}\, r!\,\delta, \tag{I,1;3}$$

when $r = 1, 2, \cdots$

Then following theorems were established by Fisher in [2], *P.* 125,

$$x_+^{-r-\frac{1}{2}} \cdot x_-^{-r-\frac{1}{2}} = \frac{(-1)^r}{2\,(2r)!}\,\pi\delta^{(2r)}(x), \tag{I,1;4}$$

when $r = 0, 1, 2, \cdots$, and, in [3], *P.* 202,

$$x^{-r} \cdot \delta^{(r-1)}(x) = \frac{(-1)^r (r-1)!}{2\,(2r-1)!}\,\delta^{(2r-1)}(x), \tag{I,1;5}$$

when $r = 1, 2, \cdots$

In the paper [4] Fisher shows that

$$\{\operatorname{sgn} x\, |x|^\lambda\} \cdot \delta(x) = 0, \tag{I,1;6}$$

for $\lambda > -1$;

$$\{\operatorname{sgn} x\, |x|^\lambda\} \cdot \delta^{(2r)}(x) = 0, \tag{I,1;7}$$

for $\lambda > 2r-1$ and $r = 0, 1, 2, \cdots$;

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$$|x|^{\lambda}\cdot\delta^{(2r+1)}(x)=0, \qquad (\mathrm{I},1;8)$$

when $\lambda>2r$ and $r=0,1,2,\cdots$;

$$|x|^{\lambda}\cdot x_{-}^{2r}=x_{-}^{2r+\lambda}, \qquad (\mathrm{I},1;9)$$

for $\lambda>-2r-2, \lambda\neq-2r-1$ and $r=0,1,2,\cdots$;

$$\{\operatorname{sgn} x|x|^{\lambda}\}\cdot x_{-}^{2r+1}=x_{-}^{2r+1+\lambda}, \qquad (\mathrm{I},1;10)$$

for $\lambda>-2r-2$, $\lambda\neq-2r-2$ and $r=0,1,2,\cdots$;

$$x_{+}^{\lambda}\cdot x_{+}^{-\lambda-1}-x_{-}^{\lambda}\cdot x_{-}^{-\lambda-1}=x^{-1}, \qquad (\mathrm{I},1;11)$$

for $\lambda\neq0, \pm1, \pm2,\cdots$;

$$(x\pm i0)^{\lambda}\cdot(x\pm i0)^{-\lambda-1}=(x\pm i0)^{-1}$$
$$=x^{-1}\mp i\pi\delta(x) \qquad (\mathrm{I},1;12)$$

when $\lambda\neq0, \pm1, \pm2,\cdots$;

$$\{\operatorname{sgn} x|x|^{\lambda}\}\cdot|x|^{-\lambda-1}=x^{-1}, \qquad (\mathrm{I},1;13)$$

for $\lambda\neq0, \pm1, \pm2, \cdots$; (of. formulae (2.1), P.317; (3.1) P.318; (3.2), P. 318; (4.1), P. 319; (4.2), P. 319; (5.1), P. 321; (6.1), P. 324 and (6.2), P.325, respectively).

In those formulae $\delta^{(r)}$ is the r-th derivative of the one-dimensional δ-measure and x_{+}^{λ}, x_{-}^{λ}, $\lambda\in C$, are the distributions defined by the formulae

$$x_{+}^{\lambda}=\begin{cases}x^{\lambda} & \text{for } x\leq0,\\ 0 & \text{for } x\leq0.\end{cases} \quad \text{and} \quad x_{-}^{\lambda}=\begin{cases}0 & \text{for } x>0,\\ |x|^{\lambda} & \text{for } x\leq0.\end{cases}$$

We extend the formulae (I, 1; 1) - (I, 1; 13) to certain kinds of distributions called "anticausal" and "causal" distributions.

I.2. We begin with some definitions. Let $x=(x_1,x_2,\cdots,x_n)$ be a point of the n-dimensional Euclidean space $\mathbf{R}^n$.

Consider a nondegenerate quadratic form in n variables of the form

$$P=P(x)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2, \qquad (\mathrm{I},2;1)$$

where $n=p+q$.

The distributions $(P\pm i0)^{\lambda}$ are defined by

$$(P\pm i0)^{\lambda}=\lim_{\varepsilon\to0}\{P\pm i\varepsilon|x|^2\}^{\lambda}, \qquad (\mathrm{I},2;2)$$

where $\varepsilon>0$, $|x|^2=x_1^2+\cdots+x_n^2$, $\lambda\in C$.

The distributions $(P\pm i0)^{\lambda}$ are analytic in λ everywhere except at $\lambda=-\frac{n}{2}-k$, $k=0,1,2,\cdots$, where they have simple poles (of. [5], P. 275). These distributions are called, respectively, "anticausal" and "causal" distributions.

Furthermore, we can write (cf. [5], formulae (2) and (2′), P. 206),

$$(P\pm i0)^{\lambda}=P_{+}^{\lambda}+e^{\pm i\pi\lambda}P_{-}^{\lambda}, \qquad (\mathrm{I},2;3)$$

where

$$P_{+}^{\lambda}=\begin{cases}P^{\lambda} & \text{for } P\geq0,\\ 0 & \text{for } P\leq0.\end{cases} \quad \text{and} \quad P_{-}^{\lambda}=\begin{cases}0 & \text{for } P\geq0,\\ (-P)^{\lambda} & \text{for } P\leq0.\end{cases} \qquad (\mathrm{I},2;4)$$

The distributions P_+^{λ} and P_-^{λ} have two sets of singularities namely $\lambda=-1, -2, \cdots, -k, \cdots$ and $\lambda=-\frac{n}{2}, -\frac{n}{2}-1, \cdots, -\frac{n}{2}-k, \cdots$.

When $\lambda=r=0,1,2,\cdots$, it follows that

$$(P\pm i0)^r=P_+^r+e^{\pm i\pi r}P_-^r=P^r \;. \tag{I,2;5}$$

We shall define (cf. [5], P. 211, formulae (7) and (8))

$$\langle\delta(P),\phi\rangle=\int\delta(P)\phi\,dx=\int_{P=0}\psi(0,u_2,\cdots,u_n)du_2\cdots du_n\;, \tag{I,2;6}$$

where

$$\psi=\phi_1(u)D\binom{x}{u} \text{ and } \phi_1(u_1,\cdots,u_n)=\phi(x_1,\cdots,x_n).$$

We write $P=u_2$ and choose the remaining u_i coordinates (with $i=2,3,\cdots,n$) arbitrarily except that the Jacobian of the x_i with respect to the u_i, which we shall denote by $D\binom{x}{u}$, fails to vanish.

Similarly we put

$$\langle\delta^{(k)}(P),\phi\rangle=\int\delta^{(k)}(P)\phi\,dx=(-1)^k\int_{P=0}\psi_{u_1}^{(k)}(0,u_2,\cdots,\cdots,u_n)du_2\cdots du_n. \tag{I,2;7}$$

Finally we observe that the following formulae are valid

$$(\operatorname{sgn} P)|P|^{\lambda}=P_+^{\lambda}-P_-^{\lambda}\;, \tag{I,2;8}$$

and

$$|P|^{\lambda}=P_+^{\lambda}+P_-^{\lambda}\;. \tag{I,2;9}$$

I.3. In this article we obtain the following results:

$P_+^{\lambda}\cdot P_-^{-1-\lambda}=-\frac{1}{2}\pi\operatorname{cosec}(\pi\lambda)\cdot\delta(P)$, when λ and $-\lambda-1\neq-\frac{n}{2}-k, n\neq2,$

$k=0,1,2,\cdots$; λ and $-\lambda-1\neq-1,-2,\cdots-k$, $k=1,2,\cdots$ (I, 3; 1)

$P_+^r\cdot\delta^{(r)}(P)=\frac{1}{2}(-1)^r r!\;\delta(P)$, when $r\in Z^+$, $n\neq2$, $r\neq\frac{n}{2}-1, \frac{n}{2}, \cdots, \frac{n}{2}+k,$

$k=-1,0,1,\cdots$ (I,3;2)

$P_-^r\cdot\delta^{(r)}(P)=\frac{1}{2}r!\;\delta(P)$, when $r\in Z^+$, $n\neq2$, $r\neq\frac{n}{2}-1, \frac{n}{2}, \cdots, \frac{n}{2}+k,$

$k=-1,0,1,\cdots$ (I,3;3)

$P_+^{-r-\frac{1}{2}}\cdot P_-^{-r-\frac{1}{2}}=\frac{(-1)^r}{2(2r)!}\pi\,\delta^{(2r)}(P)$, when $-r-\frac{1}{2}\neq-\frac{n}{2}-k$, $-2r-1\neq-\frac{n}{2}-$

k; $k=0,1,2,\cdots$; $r\in Z^+$. (I, 3; 4)

$P^{-r}\cdot\delta^{(r-1)}(P)=\frac{(-1)^r(r-1)!}{2(2r-1)!}\delta^{(2r-1)}(P)$, when $r\in Z^+$; $2r\neq\frac{n}{2}, \frac{n}{2}+1, \cdots$;

$r\neq\frac{n}{2}, \frac{n}{2}+1, \cdots$ (I,3;5)

$\{(\operatorname{sgn} P)|P|^{\lambda}\}\cdot\delta(P)=0$, when $\lambda\in C$, $\lambda\neq-k, \lambda\neq-\frac{n}{2}-k, k=0,1,2,\cdots$. (I, 3; 6)

$\{(\operatorname{sgn} P)|P|^{\lambda}\}\cdot\delta^{(2r)}(P)=0$, when $-r-\frac{1}{2}$ and $-2r-1\neq-\frac{n}{2}-k$; $k=0,1,2,$ $\cdots$; $r\in Z^+$, $\lambda\in C$. λ, $-r-\frac{1}{2}+\lambda$ and $-2r-1+\lambda\neq-1,-2,\cdots,-k$ and λ, $-r-\frac{1}{2}+\lambda$

and $-2r-1+\lambda \neq -\frac{n}{2}-k$. (I, 3; 7)

$|P|^{\lambda} \cdot \delta^{(2r+1)}(P)=0$, when $\lambda \in C$, $-r-1 \neq -\frac{n}{2}-k$, $k=0, 1, 2, \cdots$; $-2r-2 \neq -\frac{n}{2}-k$; and λ, $-r-1+\lambda \neq -k \wedge \neq -\frac{n}{2}-k$, $-2r-2+\lambda \neq -1$, $r \in Z^{+}$. (I, 3; 8)

$|P|^{\lambda} \cdot P_{+}^{2r}=P_{+}^{2r+\lambda}$, when $r \in Z^{+}$, $\lambda \neq -1, -2, \cdots, -k$; $\lambda \neq -\frac{n}{2}-k$, $k=0, 1, 2, \cdots$ (I, 3; 9)

$\{(\operatorname{sgn} P)|P|^{\lambda}\} \cdot P_{+}^{2r+1}=P_{+}^{2r+1+\lambda}$, when $r=0,1,2,\cdots$; $\lambda \in C$; $2r+\lambda+1 \neq -1$; $\lambda \neq -k \wedge -\frac{n}{2}-k$, $k \in Z^{+}$. (I, 3; 10)

$P_{+}^{\lambda} \cdot P_{+}^{-1-\lambda}-P_{-}^{\lambda} \cdot P_{-}^{-1-\lambda}=P^{-1}$, when λ and $-\lambda-1 \neq -\frac{n}{2}-k$, $n \neq 2$, $k=0, 1, \cdots$ λ and $-\lambda-1 \neq -1, -2, \cdots, -k$, $k=1,2,\cdots$ (I, 3; 11)

$(P \pm i0)^{\lambda} \cdot (P \pm i0)^{-\lambda-1}=(P \pm i0)^{-1}=P^{-1} \mp i\pi\delta(P)$, when λ and $-\lambda-1$ different from $-\frac{n}{2}-k$, $n \neq 2$, $k=0, 1, 2, \cdots$ (I, 3; 12)

$\{(\operatorname{sgn} P)|P|^{\lambda}\} \cdot |P|^{-\lambda-1}=P^{-1}$, when $\lambda \in C$, $\lambda \neq -\frac{n}{2}-k$, $k=0, 1, 2, \cdots$; $-\lambda-1 \neq -\frac{n}{2}-k$; λ and $-\lambda-1 \neq -1, -2, \cdots, -k$; $n \neq 2$. (I, 3; 13)

I.4. First we shall prove formulae (I, 3; 1) and (I, 3; 11).

Taking into account formula (I, 2; 3), we have ($\lambda \in C$)

$$(P+i0)^{\lambda}=P_{+}^{\lambda}+e^{i\pi\lambda}P_{-}^{\lambda}, \tag{I, 4; 1}$$

and

$$(P+i0)^{-\lambda-1}=P_{+}^{-\lambda-1}+e^{i\pi(-\lambda-1)}P_{-}^{-\lambda-1}. \tag{I, 4; 2}$$

By multiplying the left-hand members of (I, 4; 1) and (I, 4; 2), one verifies that

$$(P+i0)^{\lambda} \cdot (P+i0)^{-\lambda-1}=(P+i0)^{-1}. \tag{I, 4; 3}$$

This last formula is valid when λ and $-\lambda-1$ are complex numbers different from $-\frac{n}{2}-k$, $k=0,1,2,\cdots$ and $n \neq 2$ (of. [6], Theorem 9, P. 28).

.We have also, from formula (4, 6), P. 576, [7], for $n \neq 2$,

$$(P+i0)^{-1}=\frac{1}{P}-i\pi\delta(P). \tag{I, 4; 4}$$

From (I, 4; 3) and (I, 4; 4) we get

$$(P+i0)^{\lambda} \cdot (P+i0)^{-\lambda-1}=\frac{1}{P}-i\pi\delta(P), \tag{I, 4; 5}$$

λ and $-\lambda-1$ different from $-\frac{n}{2}-k$, $k=0, 1, 2, \cdots, n \neq 2$.

We obtain, by multiplying the right-hand members of (I, 4; 1) and (I, 4; 2), the formula

$$\{P_{+}^{\lambda}+e^{i\pi\lambda}P_{-}^{\lambda}\} \cdot \{P_{+}^{-\lambda-1}+e^{i\pi(-\lambda-1)}P_{-}^{-\lambda-1}\}$$

$$=P_{+}^{\lambda} \cdot P_{+}^{-\lambda-1}-P_{-}^{\lambda} \cdot P_{-}^{-\lambda-1}+\{2i \operatorname{sen} \pi\lambda\} P_{-}^{\lambda} \cdot P_{+}^{-\lambda-1}, \tag{I, 4; 6}$$

valid when λ and $-\lambda-1$ different from $-\frac{n}{2}-k$ and different from $-1, -2, \cdots, -k$, $k \in Z^{+}$.

Note. The "heterodox" multiplicative products which appear in (I, 4; 6) and the other products in what follows will be justified in paragraph I.14.

From (I, 4; 5) and (I, 4; 6) we get

$$\frac{1}{P}-i\pi\,\delta(P)=P_{+}^{\lambda}\cdot P_{+}^{-\lambda-1}-P_{-}^{\lambda}\cdot P_{-}^{-\lambda-1}+\{2i \text{ sen } \pi\lambda\}P_{+}^{-\lambda-1}\cdot P_{-}^{\lambda} \qquad (I,4;7)$$

when λ and $-\lambda-1\neq-\frac{n}{2}-k$, $n\neq 2$; $k=0,1,2,\cdots$, λ and $-\lambda-1\neq-1,-2,\cdots,-k$, $k=1,2,\cdots$

It is immediately seen by equalizing the real and the imaginary parts of both members of (I, 4; 7), that

$$P_{+}^{\lambda}\cdot P_{+}^{-\lambda-1}-P_{-}^{\lambda}\cdot P_{-}^{-\lambda-1}=\frac{1}{P}, \qquad (I,4;8)$$

when λ and $-\lambda-1\neq-\frac{n}{2}-k$, $n\neq 2$, $k=0,1,2,\cdots$; λ and $-\lambda-1\neq-1,-2,\cdots,-k$, $k=1,2,\cdots$ and

$$\{2i \text{ sen } \pi\lambda\}P_{+}^{-\lambda-1}\cdot P_{-}^{\lambda}=-i\,\pi\delta(P);$$

or, equivalently,

$$P_{+}^{-\lambda-1}\cdot P_{-}^{\lambda}=-\frac{1}{2}\pi \text{ cosec }(\pi\lambda)\ \delta(P), \qquad (I,4;9)$$

when λ and $-\lambda-1\neq-\frac{n}{2}-k$, $n\neq 2$, $k=0,1,2,\cdots$; λ and $-\lambda-1\neq-1,-2,\cdots,-k$, $k=1,2,\cdots$

Formulae (I, 4; 8) and (I, 4; 9) are identical with formulae (I, 3; 11) and (I, 3; 1). ∎

I.5. In this paragraph we shall prove formulae (I, 3; 2) and (I, 3; 3).

We begin by considering the formula (cf. [7], P. 577, formula (4.9))

$$(P+i0)^{-(r+1)}=\frac{1}{P^{r+1}}+\frac{(-1)^{r+1}}{r!}\pi i\ \delta^{(r)}(P), \qquad (I,5;1)$$

for $r=0,1,2,\cdots$, $r\neq\frac{n}{2}-1,\ \frac{n}{2},\ \cdots,\ \frac{n}{2}+k$; $k=-1,0,1,2,\cdots$

Taking into account (I, 4; 5), with $\lambda=r$, we obtain

$$(P+i0)^{r}(P+i0)^{-(r+1)}=(P+i0)^{-1}=\frac{1}{P}-i\pi\delta(P), \qquad (I,5;2)$$

where $r=0,1,2,\cdots$; $-r-1\neq-\frac{n}{2}-k$, $n\neq 2$, $k=0,1,2,\cdots$

From the right-hand member of (I, 4; 1), when $\lambda=r$, and (I, 5; 1) we obtain, taking into account formula (I, 2; 5),

$$\{P_{+}^{r}+e^{i\pi r}P_{-}^{r}\}\cdot\{P^{-(r+1)}+\frac{(-1)^{r+1}}{r!}\pi i\delta^{(r)}(P)\}=P^{r}\cdot P^{-(r+1)}+\frac{(-1)^{r+1}}{r!}\pi i\delta^{(r)}(P)\cdot P^{r}, \qquad (I,5;3)$$

for $r=0,1,2,\cdots$, $r\neq\frac{n}{2}-1,\ \frac{n}{2},\ \cdots,\ \frac{n}{2}+k$, $k=-1,0,1,2,\cdots$

We get, taking into account the imaginary parts of the right hand members of (I, 5; 2) and (I, 5; 3),

$$P^{r}\ \delta^{(r)}(P)=(-1)^{r}\ r!\ \delta(P), \qquad (I,5;4)$$

for $r=0,1,2,\cdots;\ r\neq\frac{n}{2}-1,\ \frac{n}{2},\cdots,\frac{n}{2}+k,\ k=-1,0,1,2,\cdots$

We know that for r a nonnegative integer,

$$P^r=P^r_+ +(-1)^r P^r_- . \qquad (\mathrm{I},5;5)$$

Then, from ((I,5;4) and (I,5;5), we get

$$P^r_+\cdot\delta^{(r)}(P)=\frac{1}{2}(-1)^r r!\ \delta(P), \qquad (\mathrm{I},5;6)$$

for $r\in Z^+,\ n\neq 2,\ r\neq\frac{n}{2}-1,\ \frac{n}{2},\cdots,\frac{n}{2}+k,\ k=-1,0,1,\cdots$, analogously we get

$$P^r_-\cdot\delta^{(r)}(P)=\frac{1}{2}r!\ \delta(P), \qquad (\mathrm{I},5;7)$$

for $r\in Z^+,\ n\neq 2,\ r\neq 2,\ r\neq\frac{n}{2}-1,\ \frac{n}{2},\cdots,\frac{n}{2}+k,\ k=-1,0,1,\cdots$

Formulae (I,5;6) and (I,5;7) are identical with formulae (I,3;2) and (I,3;3). ∎

I.6. Formula (I,2;3) for $\lambda=-r-\frac{1}{2},\ r=0,1,2,\cdots$, reads

$$(P\pm i0)^{-r-\frac{1}{2}}=P_+^{-r-\frac{1}{2}}+e^{\pm i\pi(-r-\frac{1}{2})}P_-^{-r-\frac{1}{2}}. \qquad (\mathrm{I},6;1)$$

Therefore, we obtain

$$\{(P\pm i0)^{-r-\frac{1}{2}}\}^2=\{P_+^{-r-\frac{1}{2}}\}^2+\{e^{\pm i\pi(-r-\frac{1}{2})}P_-^{-r-\frac{1}{2}}\}^2+2P_+^{-r-\frac{1}{2}}e^{\pm i\pi(-r-\frac{1}{2})}P_-^{-r-\frac{1}{2}} \qquad (\mathrm{I},6;2)$$

for $r=0,1,2,\cdots;\ -r-\frac{1}{2}\neq-\frac{n}{2}-k,\ k=0,1,2,\cdots;\ -2r-1\neq-\frac{n}{2}-k$.

The left-hand member of (I,6;2) reads (of. [6], P.28, Theorem 9 and [7], P.577, formula (4.9)),

$$\{(P\pm i0)^{-r-\frac{1}{2}}\}^2=(P\pm i0)^{-r-\frac{1}{2}}(P\pm i0)^{-r-\frac{1}{2}}=(P\pm i0)^{-(2r+1)}=$$
$$=P^{-(2r+1)}\mp\frac{i\pi(-1)^{2r}}{(2r)!}\delta^{(2r)}(P), \qquad (\mathrm{I},6;3)$$

when $-r-\frac{1}{2}\neq-\frac{n}{2}-k;\ -2r-1\neq-\frac{n}{2}-k;\ k=0,1,2,\cdots$; and $r\in Z^+$.

Thking into account the imaginary parts of the right-hand members of (I,6;2) and (I,6;3), we obtain

$$P_+^{-r-\frac{1}{2}}P_-^{-r-\frac{1}{2}}=\frac{(-1)}{2(2r)!}\pi\delta^{(2r)}(P), \qquad (\mathrm{I},6;4)$$

for $-r-\frac{1}{2}\neq-\frac{n}{2}-k;\ -2r-1\neq-\frac{n}{2}-k;\ k=0,1,2,\cdots$, and $r\in Z^+$.

Formula (I,6;4) is identical with fomula (I,3;4). ∎

I.7. Now we shall obtain formula (I,3;5). From (I,5;1) we get

$$(P+i0)^{-2r}=\mathrm{Pf}P^{-2r}-\frac{(-1)^{2r-1}}{(2r-1)!}i\pi\delta^{(2r-1)}(P), \qquad (\mathrm{I},7;1)$$

for $r\in Z^+,\ 2r\neq\frac{n}{2},\ \frac{n}{2}+1,\cdots$

Also we have

$$(P\pm i0)^{-2r}=[(P\pm i0)^{-r}]^2=(P\pm i0)^{-r}(P\pm i0)^{-r}=\left\{\mathrm{Pf}\ \frac{1}{Pr}\right\}^2-$$

$$-\frac{\pi^2}{\{(r-1)!\}^2}\{\delta^{(r-1)}(P)\}^2\mp 2\left\{\mathrm{Pf}\ \frac{1}{P^r}\right\}\left\{\frac{(-1)^r}{(r-1)!}i\pi\delta^{(r-1)}(P)\right\}, \quad (\mathrm{I},7;2)$$

when $r\in Z^+$; $-r$ and $-2r\neq -\frac{n}{2}-k$, $k=0,1,\cdots$

By comparing the real and imaginary parts of the right-hand members of (I, 7; 2), and (I, 7; 2), we get

$$\mathrm{Pf}\ P^{-2r}=\left\{\mathrm{Pf}\ \frac{1}{P^r}\right\}^2-\frac{\pi^2}{[(r-1)!]^2}\{\delta^{(r-1)}(P)\}^2, \quad (\mathrm{I},7;3)$$

and

$$\mathrm{Pf}\ \frac{1}{P^r}\delta^{(r-1)}(P)=\frac{1}{2}(-1)^r\frac{(r-1)!}{(2r-1)!}\delta^{(2r-1)}(P), \quad (\mathrm{I},7;4))$$

for $r\in Z^+$; $2r\neq\frac{n}{2}$, $\frac{n}{2}+1$, $\cdots$; $r\neq\frac{n}{2}$, $\frac{n}{2}+1$, $\cdots$

Formula (I, 7; 4) is a generalization of the formula $\mathrm{p}\nabla\frac{1}{x}\cdot\delta(x)=-\frac{1}{2}\delta'(x)$ due to A. González Domínguez and R. Scarfiello (cf. [8]) and is identical with formula (8.1), P. 17, [9] ■

I.8. From (I, 3; 4) we get

$$\delta^{(2r)}(P)=(-1)^r 2\,(2r)!\ \pi P_+^{-(r+\frac{1}{2})}P_-^{-(r+\frac{1}{2})}, \quad (\mathrm{I},8;1)$$

for $-r-\frac{1}{2}\neq-\frac{n}{2}-k$; $-2r-1\neq-\frac{n}{2}-k$; $k=0,1,2,\cdots$; and $r\in Z^+$.

The following formula is also valid ($\lambda\in C$),

$$(\mathrm{sgn}\ P)\,|P|^\lambda=P_+^\lambda-P_-^\lambda. \quad (\mathrm{I},8;2)$$

By multiplying the left and the right-hand members of the preceding formulae, we get

$$\{(\mathrm{sgn}\ P)\,|P|^\lambda\}\cdot\delta^{(2r)}(P)=(-1)^r 2\,(2r)!\,\pi\cdot\{P_+^{-r-\frac{1}{2}}P_-^{-r-\frac{1}{2}}\}\{P_+^\lambda-P_-^\lambda\}=0, \quad (\mathrm{I},8;3)$$

for $-r-\frac{1}{2}$ and $-2r-1\neq-\frac{n}{2}-k$, $k=0,1,2,\cdots$; and $r\in Z^+$; $\lambda\in C$; λ, $-r-\frac{1}{2}+\lambda$ and $-2r-1+\lambda\notin Z^-$ and λ, $-r-\frac{1}{2}+\lambda$ and $-2r-1+\lambda\neq-\frac{n}{2}-k$.

The last formula is identical with (I, 3; 7). ■

If we put $r=0$ in (I, 8; 4), we get formula (I, 3; 6). ■

I.9. We shall establish now the formula (I, 3; 8).

From (I, 3; 4) we have, for $-r-1\neq-\frac{n}{2}-k$, $k=0,1,2,\cdots$; $-2r-2\neq-\frac{n}{2}-k$; $r\in Z^+$

$$\delta^{(2r+1)}(P)=((-1)^{r+\frac{1}{2}}2\,(2r+1)!\ \pi P_+^{-r-1}P_-^{-r-1}. \quad (\mathrm{I},9;1)$$

We know also that ($\lambda\in C$)

$$|P|^\lambda=P_+^\lambda+P_-^\lambda. \quad (\mathrm{I},9;2)$$

Multiplying the two last equalities, we obtain

$$|P|^{\lambda}\cdot\delta^{(2r+1)}(P)=0, \qquad (I,9;3)$$

when $\lambda\epsilon C$; $-r-1\neq-\frac{n}{2}-k$, $k=0,1,2,\cdots$; $-2r-2\neq-\frac{n}{2}-k$; and λ, $-r-1+\lambda$

$\neq-k\wedge\neq-\frac{n}{2}-k$, $-2r-2+\lambda\neq-1$, $r\epsilon Z^{+}$.

Formula (I,9;3) proves our thesis (I,3;8). ■

I.10. We shall obtain the formula (I,3;9).

First, we register the formula (cf. [6], P. 28, Theorem 9)

$$(P+i0)^{2r}\cdot(P+i0)^{\lambda}=(P+i0)^{2r+\lambda}, \qquad (I,10;1)$$

when λ and $2r+\lambda$ are complex numbers different from $-\frac{n}{2}-k, r, k=0,1,2,\cdots$

From (I,4;1) and (I,9;2) we have

$$(P+i0)^{\lambda}=(|P|^{\lambda}-P_{-}^{\lambda})+e^{i\pi\lambda}P_{-}^{\lambda}, \qquad (I,10;2)$$

for $\lambda\epsilon C$, and

$$(P+i0)^{2r}=P_{+}^{2r}+e^{i2\pi r}P_{-}^{2r}, \qquad (I,10;3)$$

for $r=0,1,2,\cdots$

Multiplying the left-hand members of (I,10;2) and (I,10;3) we obtain

$$(P+i0)^{\lambda}(P+i0)^{2r}=(P+i0)^{\lambda+2r}=P_{+}^{\lambda+2r}+e^{i\pi(\lambda+2r)}P_{-}^{\lambda+2r} \qquad (I,10;4)$$

for $\lambda\neq-\frac{n}{2}-k$; $k=0,1,2,\cdots$ and $r\epsilon Z^{+}$.

By multiplying the right-hand members of (I,10;2) and (I,10;3), we obtain

$$[(|P|^{\lambda}-P_{-}^{\lambda})+e^{i\pi\lambda}P_{-}^{\lambda}]\cdot[P_{+}^{2r}+e^{i2\pi r}P_{-}^{2r}]=$$
$$=|P|^{\lambda}P_{+}^{2r}+e^{i2\pi r}|P|^{\lambda}P_{-}^{2r}-(e^{i2\pi r}-e^{i\pi(\lambda+2r)})P_{-}^{\lambda+2r}, \qquad (I,10;5)$$

for $\lambda\epsilon C$; $r=0,1,2,\cdots$; $\lambda\neq-1,\cdots,-k$; $\lambda\neq-\frac{n}{2}-k$ and $k=0,1,2,\cdots$

From (I,10;4) and (I,10;5) we conclude

$$|P|^{\lambda}\cdot P_{+}^{2r}=P_{+}^{2r+\lambda}, \qquad (I,10;6)$$

when $r\epsilon Z^{+}$, $\lambda\neq-1,-2,\cdots,-k$; $\lambda\neq-\frac{n}{2}-k$, $k=0,1,2,\cdots$ ■

I.11. Taking into account formula (I,2;8) we have, for $\lambda\epsilon C$,

$$(\mathrm{sgn}\ P)|P|^{\lambda}=P_{+}^{\lambda}-P_{-}^{\lambda}. \qquad (I,11;1)$$

Then

$$\{(\mathrm{sgn}\ P)|P|^{\lambda}\}\cdot P_{+}^{2r+1}=\{P_{+}^{\lambda}-P_{-}^{\lambda}\}\cdot P_{+}^{2r+1}=P_{+}^{\lambda+2r+1}, \qquad (I,11;2)$$

when $r=0,1,2,\cdots$; $\lambda\epsilon C$ and $\lambda+2r+1\neq-1$;

$\lambda\neq-k\wedge\neq-\frac{n}{2}-k$, $k\epsilon Z^{+}$.

The last formula is identical with (I,3;10). ■

I.12. Formula (I,3;12) is a particular case of the following formula (cf. [6], p. 23, formula (I,3;17))

$$(P\pm i0)^{\lambda}\cdot(P\pm i0)^{\mu}=(P\pm i0)^{\lambda+\mu}, \qquad (I,12;1)$$

valid for all λ, μ and $\lambda+\mu \epsilon$ C and different from $-\frac{n}{2}-k$, $k=0, 1, 2, \cdots$

Therefore

$$(P\pm i0)^{\lambda}\cdot(P\pm i0)^{-\lambda-1}=(P\pm i0)^{-1},$$

when λ and $-\lambda-1$ different from $-\frac{n}{2}-k$, $n\neq 2$, $k=0, 1, 2, \cdots$

The second equality of the formula (I, 3; 12) is due to D. Bresters [7], *P*. 577, formula (4.9).

I. 13. Finally, we shall prove the formula (I, 3; 13).

We have, taking into account the (I, 2; 8), (I, 2; 9) and (I, 3; 11)

$$\{(\operatorname{sgn} P)|P|^{\lambda}\}\cdot|P|^{-\lambda-1}=P^{-1}, \qquad \text{(I, 1; 31)}$$

which is valid for $\lambda \epsilon C$, $\lambda\neq-\frac{n}{2}-k$, $k=0, 1, 2, \cdots$; $-\lambda-1\neq-\frac{n}{2}-k$; and λ and $-\lambda-1\neq-1, -2, \cdots -k$; $n\neq 2$ ∎

I. 14. If S and T are distributions we shall define theine multiplicative product by the formula

$$S\cdot T=\lim_{n\to\infty}\{S*g_n(x)\}\cdot\{T*g_n(x)\} \qquad \text{(I, 14; 1)}$$

if the limit exists for every mollifier $g_n(x)$. The symbol $*$ denotes, as usual, convolution. By a mollifier we mean a sequence $g_n(x)=ng(nx)$, where the function $g(x)$ has the propeperties

1) $g(x)\geq 0$, 2) $g \epsilon C_0^{\infty}$, 3) $\int_{\infty}^{\infty} g(x)dx=1$, 4) $g(xx)=g(-x)$, 5) supp $g(x)=[-1, 1]$, 6) $g(x)$ is increasing for $-1\leq x\leq 0$ and decreasing for $0\leq x\leq 1$.

In order to give a sense to all products evaluated in the paragraphs I. 1 to I. 13 (or "regularize" them) we deal with the multidimensional generalizations obtained by means of a change of variable

Let ϕ_s be a distribution of one variable s and let $u(x) \epsilon C^{\infty}(\mathbf{R}_n)$ be such that the $(n-1)$-dimensional manifold $u(x_1, x_2, \cdots, x_n)=0$ has no critical point; $\phi_{u(x)}$ denotes the distribution defined on $\mathbf{R}_n$ by the formula (called the Leray formula, (cf. [10], *p*. 102))

$$\int_{\mathbf{R}^n}\phi_{u(x)}f(x)\,dx_1\cdots dx_n=\int_{-\infty}^{\infty}\phi_s\,ds\int_{u(x)=s}f(x)\,w_u(x, dx); \qquad \text{(I, 14; 2)}$$

here w_u is an $(n-1)$-dimensional form on u defined as follows:

$$du\wedge dw=dx_1\wedge dx_2\wedge\cdots\wedge dx_n;$$

the manifold $u(x)=s$ has the orientation such that $w_u(x, dx)>0$.

We remark that the one-dimensional products we have considered, when applying Leray's formula are due to A. Brédimas [11] and B. Fisher [1], [4].

We remit, mainly, to González Domínguez's Note (of. [9], especially Theorems 8, 9, 10, 11 and 12).

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