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On the Solution of the Operator Equation AT - TB = S*

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Let H, K be two Hilbert spaces over complex field Λ. Let B(H, K) denote the set of all bounded linear operators from H to K. If H = K, we write B(H) instead of B(H, K). Consider the operator defined by the equation

$$D_{AB}T = AT - TB$$

which is called generalized derivation. In [1] it was shown that $\sigma(D) \subset \sigma(A) - \sigma(B)$ where $\sigma(T)$ denotes the spectrum of T. Therefore if $\sigma(A) \cap \sigma(B) = \emptyset$, then D_{AB} is invertible and hence the equation

$$AT - TB = S \qquad (1)$$

has an unique solution for every $S \in B(K, H)$. If $O \in \sigma(D_{AB})$, then the solution is complicated. In the present note we confine our attention to the solution of (1) in a special case: A and B are nilpotent operators, $S \in \overline{R(D_{AB})}$ (the closure of the range of D_{AB}). By a program well designed we shall show that (1) dose have solutions under some sufficient conditions. Obviously, such a result can be viewed as a sufficient condition for D_{AB} to have closed range.

For nilpotent operators A and B of order n, define subspaces

$$H_i = \ker A^i \ominus \ker A^{i-1}, \quad K_i = \ker B^i \ominus \ker B^{i-1}, \qquad 1 \le i \le n, \qquad (2)$$

let P_i, Q_i be the orthogonal projections of H, K on H_i, K_i respectively and

$$A_i = P_i A P_{i+1}, \quad B_i = Q_i B Q_{i+1}, \quad 1 \le i < n. \qquad (3)$$

We begin with stating two definitions. (For details see [2].)

Definition 1. Let $1 \le k_1 < \cdots < k_{m-1} < n$ be $m-1$ natural numbers. If only $R(A^{k_p})$, $1 \le p \le m-1$ are closed, then we say that the C-state of A is $\{k_1, \cdots, k_{m-1}\}$ and denote this situation by $C(A) = \{k_1, \cdots, k_{m-1}\}$. If all $R(A_k)$ are closed, but only A_{k_p}, $1 \le p \le m-1$ are not invertible (equivalently $R(A_{k_p}) \subsetneq H_{k_p}$), then we say that the I-state of A is $\{k_1, \cdots, k_{m-1}\}$ and denote this situation by $I(A) = \{k_1, \cdots, k_{m-1}\}$.

Let $X_i(p)$, $1 \le i \le k_p$, $1 \le p \le m$ be pairwise disjoint Banach spaces and X be the direct sum

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$$X=\sum_{p=1}^{m}\sum_{i=1}^{k_p}X_i(p).$$

then every $A\in B(X)$ has the form

$A=(A)=(a_{ij}(p,q))\quad 1\le p,\ q\le m,\ 1\le i\le k_p,\ 1\le j\le k_q,$

where $a_{ij}(p,q)\in B(X_j(q),X_i(p))$.

Definition 2. If $a_{ij}(p,q)$ satisfy the conditions

i) $a_{ij}(p,p)=0$ for $i\le j$ and $a_{ij}(p,q)=0$ for $p\ne q$ and $i\ge j-1$;

ii) $a_{i,i-1}(p,p)$ are invertible for $1\le i<k_p$,

then (A) is called a matrix of type $I\{k_1,\cdots,k_{m-1}\}$. If ii) is replaced by

ii′) $a_{i,i-1}(p,p)$ are injective for $1\le i<k_p$ and the frist k_{p-1} ones are invertible,

then (A) is called a matrix of type $C\{k_1,\cdots,k_{m-1}\}$.

In [2] we have obtained

Theorem 1. Let H be a Hilbert space, $A\in B(H)$ be a nilpotent operator of order n, and $I(A)=(C(A)\supset)\{k_1,\cdots,k_{m-1}\}$, then A has a matrix form of type $I\{k_1,\cdots,k_{m-1}\}(C\{k_1,\cdots k_{m-1}\})$ which will be called the fine matrix representation of A.

Proposition 1. Let H, K be complex Hilbert spaces, $A\in B(H)$, $B\in B(K)$ be nilpotent operators of order n, if $I(A)=\varnothing$, then the equation (1) has solutions for $S\in\overline{R(D_{AB})}$.

Proof. Since $S\in\overline{R(D_{AB})}$, there exists a sequence $\{T^{(k)}\}\subset B(K,H)$ such that

$$AT^{(k)}-T^{(k)}B\to S \tag{4}$$

Define H_i, K_i, A_i, B_i by (2) and (3), then A, B, $T^{(k)}$, S have the following matrix forms

$$A=\begin{pmatrix}0 & A_1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ & & & A_{n-1} \\ 0 & \cdots & \cdots & 0\end{pmatrix}\qquad B=\begin{pmatrix}0 & B_1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ & & & B_{n-1} \\ 0 & \cdots & \cdots & 0\end{pmatrix}$$

$$T^{(k)}=(t_{ij}^{(k)}),\quad S=(S_{ij})$$

where $A_1,\cdots,A_{n-1}$ are invertible. Frist we consider column 1 of (4):

$$A_1t_{21}^{(k)}+*t_{31}^{(k)}+\cdots+*t_{n1}^{(k)}\to S_{11},\ \cdots,\ A_{n-1}t_{n1}^{(k)}\to S_{n-1,1},0\to S_{n,1}. \tag{5}$$

Since A_k are invertible, the solution of (1) is

$t_{n1}=A_{n-1}^{-1}S_{n-1\,1},\ \cdots\quad t_{21}=A_1^{-1}(S_{11}-*t_{31}-\cdots-*t_{n1})$

and we observe that

$$t_{i1}^{(k)}\to t_{i1}\quad\text{for } 2\le i\le n. \tag{6}$$

The column 2 of (4) is $A_1t_{22}^{(k)}+\cdots+*t_{n2}^{(k)}-t_{11}^{(k)}B_2\to S_{12},\ \cdots,\ A_{n-1}t_{n2}^{(k)}-t_{n-1,1}^{(k)}B_2\to S_{n-1,2}$, $-t_{n1}^{(k)}B_2\to S_{n2}$. (6) implies $-t_{n1}B_2=S_{n2}$. Since $t_{n1},\cdots,t_{21}$ have been known, $t_{n2},\cdots,t_{32}$ can be determined. Continuinuing in the same way we obtain the entries $\{t_{ij}, i \quad j\}$ and

$$t_{ij}^{(k)} \to t_{ij} \quad \text{for} \quad i > j. \tag{7}$$

To find the others, we define arbitrarily t_{1j} for $1 \le j \le n$ and consider again the columns $2, \cdots, n$ in turn. Using the invertibility of A_i's we can find the entries $\{t_{ij}, \ i \le j\}$. Finally put $T = (t_{ij})$ which is a solution of (1).

Remark 1. Since t_{1j} are arbitrary for $1 \le j \le n$, the solution is not unique.

Remark 2. (7) implies that the equations determined by the last line are automatically satisfied.

Proposition 2. Let H, K be complex Hilbert spaces, $A \in B(\mathrm{H})$, $B \in B(\mathrm{K})$ be nilpotent operators of order n. If $\mathrm{I}(A) \subset \mathrm{C}(B)$, then the equation (1) has solutions for $S \in \overline{\mathrm{R}(D_{AB})}$.

Proof. By Theorem 1 we may assume that

$$A = (A) = (a_{ij}(p, q)), \ B = (B) = (b_{ij}(p, q))$$

where $a_{ij}(p, q)$, $b_{ij}(p, q)$ satisfy the conditions Ⅰ), Ⅱ), and Ⅰ), Ⅱ$'$) respectively.

The solution is rather complicated which depends partly on the invertibility of $a_{i,i+1}(p,p)'$s and partly on the closedness of $\mathrm{R}(B^{kp})'$s. By the proof of Proposition 1 we can consider directly the equation (1) instead of (4). For convenience's sake, we introduce some notation frist. The col $j(q)$ of (S) is the column

$$(S_{1j}(1, q), \cdots, S_{k_1j}(1, q), \cdots S_{1j}(m, q), \cdots, S_{k_mj}(m, q))^{\mathrm{T}};$$

the line $i(p)$ of (S) is the line

$$(S_{i1}(p, 1), \cdots, S_{ik_1}(p, 1), \cdots, S_{i1}(p, m), \cdots, S_{ik_m}(p, m));$$

the equ $\boldsymbol{ij(p,q)}$ of (1) is the one determined by the entry with subscript $ij(p, q)$ of (1), namely

$$\begin{aligned}&\left(\sum_{\substack{i+2 \le j' \le k_{q'} \\ 1 \le q' < p}} + \sum_{\substack{i+1 \le j' \le k_p \\ q' = p}} + \sum_{\substack{i+2 \le j' \le k_{q'} \\ p < q' \le m}}\right) a_{ij'}(p, q')t_{j'j}(q', q) \\ &\left(-\sum_{\substack{1 \le j' \le j-2 \\ 1 \le q' < q}} - \sum_{\substack{1 \le j' \le j-1 \\ q' = p}} - \sum_{\substack{1 \le j' \le j-2 \\ p < q' \le m}}\right) t_{ij'}(p, q')b_{j'j}(q', q) = S_{ij}(p, q).\end{aligned} \tag{8}$$

Remark 2 implies that we need not consider the equ's with such subscripts $ij(p, q)$ where $i = k_p$, $p \ge q$, $1 \le q \le m$.

The whole solution contains several steps;

1) Consider the col's $1(q)$ $(1 \le q \le m)$ of (1). Since $b_{i1}(p, q) = 0$ for $1 \le i \le k_p$, $1 \le p, q \le m$, the sum of $t_{ij'}b_{j'j}$ in (8) are zero, consequently the corresponding equ's are the simpliest. Frist we solve the equ's $i1(p, q)$,

$k_{m-1}-1 \le i < k_m$. By the condition Ⅰ) they can be written as

$$a_{k_{m-1}-1,k_{m-1}}(m,m)t_{k_{m-1},1}(m,q)+\cdots+* t_{k_m,1}(m,q)=S_{k_{m-1}-1,1}(m,q),$$

$$\cdots,\ a_{k_m-1,k_m}(m,m)t_{k_m,1}(m,q)=S_{k_m-1,1}(m,q)$$

By condition Ⅱ) we can find the solution $\{t_{i1}(m,q),\ k_{m-1}\le i\le k_m\}$. Next we turn to the equ $k_{m-1}-1,1(m-1,q)$. Since $k_p < k_{p+1}$, the frist sum in (8) is zero. Thus the equ $k_{m-1}-1,1(m-1,q)$ has the form

$$a_{k_{m-1}-1,k_{m-1}}(m-1,m-1)t_{k_{m-1},1}(m-1,q)+\sum_{k_{m-1}<j'\le k_m} a_{k_{m-1}-1,j'}(m-1,m)t_{j'1}(m,q)$$
$$=S_{k_{m-1}-1,1}(m-1,q).$$

Since all of t_{ij} in the above sum are known, by Ⅱ) we can obtain $t_{k_{m-1}1}(m-1,q)$. Put $i=k_{m-1}-2,\cdots,1$ in turn and for each i consider the equ's $i1(p,q)$ for $i<k_p$. Using the same argument we can find the entries $\{t_{i1}(p,q),\ 2\le i\le k_p,\ 1\le p,\ q\le m\}$.

2) consider the col's $2(q)$ $(1\le q\le m)$. In this case by Ⅰ) only $t_{i1}(p,q)$ $b_{12}(q,q)$ appear in the left side of the equ's $i2(p,q)$. (See equ (8).) However the entries $\{t_{i1}(p,q),\ 2\le i\le k_p,\ 1\le p,q\le m\}$ have been known we can solve the equ' $i2(p,q)$, $2\le i<k_p$ to obtain the entries $\{t_{i2}(p,q),\ 3\le i\le k_p,\ 1\le p,\ q\le m\}$. Continuing in the same way we obtain the entries $\{t_{ij}(p,q),\ i>j,\ 1\le p,q\le m\}$. Moreover we have the following set equality

$\{j, a_{ij}(p,q)\ (b_{ij}(p,q))$ may be nonzero$\}-\{j,\ t_{j\,i+1}(p,p')$ are determined$\} = \begin{cases} \emptyset, & q\ne p \\ \{i+1\}, & q=p \end{cases}$ K_q, for any given $i,\ p,q,q'$.

3) To solve the equ's $ij(m,q)$ $(i<j\le k_q,\ 1\le q\le m)$ we define arbitrarily $t_{1j}(m,q)$ for $1\le j\le k_q$, $1\le q\le m$. By Ⅱ) we can find the solution $\{t_{ij}(m,q),\ i\le j\le k_q,\ 1\le q\le m\}$ according to the order $j=2,\cdots,k_m$, for each j let $q=1,\cdots,m$ and $i=j-1,\cdots,1$. Thus all entries in the line's $i(m)$, $1\le i\le k_m$ are determined.

4) In order to determine the entries $t_{ij}(m-1,q)$, we can not follow 3) word for word. The reason is that if define $t_{1j}(m-1,q)$ arbitrily then $t_{k_{m-1},k_m-1}(m-1,m)$ will be determined by the equ$k_{m-1}-1,k_m-1$ $(m-1,m)$ (via $a_{k_{m-1}-1\ k_{m-1}}(m-1,m-1)$ is invertible) and it will appear in the equ $k_{m-1}k_m(m-1,m)$, because, by the definition of $K_{k_m}(m)$ $b_{k_m-1,k_m}(m,m)\ne 0$. But it is not sure that the entry will satisfy the equation. While the analysis is done, the solution is at hand. Consider the equ's $k_{m-1}k_{m-1}+i(m-1,m)$, $1\le i\le k_m-k_{m-1}$, which can be written as

$$\left(-\sum_{q=m}\sum_{j=k_{m-1}}^{k_{m-1}+i-2} t_{k_{m-1}j}(m-1,q)b_{j,k_{m-1}+i}(q,m)-\sum_{j=k_{m-1}}^{k_{m-1}+i-1} t_{k_{m-1},j}(m-1,m)b_{j,k_{m-1}+i}(m,m)\right.$$

$$=S'_{k_{m-1},k_{m-1}+i}(m-1,m) \tag{10}$$

where

$$S'_{k_{m-1},k_{m-1}+i}(m-1,m)=S_{k_{m-1},k_{m-1}+i}(m-1,m)-\sum_{j=k_{m-1}+2}^{k_m} a_{k_{m-1},j}(m-1,m)t_{j,k_{m-1}+i}$$

$$(m,m)+\sum_{q=1}^{m}\sum_{j=1}^{k_{m-1}-1} t_{k_{m-1}j}(m-1,q)b_{j,k_{m-1}+i}(q,m)$$

are known. For each entry $t_{k_{m-1},j}(m-1,q)$ in the left side of (10), consider equ's $i, i+j-k_{m-1}+1\,(m-1,q)$, $1\leq i<k_{m-1}$. which can be written as

$$a_{12}(m-1,m-1)t_{2,j-k_{m-1}+2}(m-1,q)-t_{1,k_{m-1}+1}(m-1,q)b_{j-k_{m-1}+1,j-k_{m-1}+2}$$

$$\cdot(q\cdot q)=S'_{1,j-k_{m-1}+2}(m-1,q),\cdots, \tag{11}$$

$$a_{k_{m-1}-1,k_{m-1}}(m-1,m-1)t_{k_{m-1},j}(m-1,q)-t_{k_{m-1}-1,j-1}(m-1,q)b_{j-1,j}(q,q)$$

$$=S'_{k_{m-1}-1,j}(m-1,q)$$

where

$$S'_{i,i+j-k_{m-1}+1}(m-1,q)=S_{i,i+j-k_{m-1}+1}(m-1,q)-\sum_{q'=1}^{m}\sum_{j'=i+2}^{k_{q'}} a_{ij'}(m-1,q')$$

$$\cdot t_{j',i+j-k_{m-1}+1}(q',q)+\sum_{q'=1}^{m}\sum_{j'=1}^{i+j-k_{m-1}-1} t_{ij'}(m-1,q')b_{j',i+j-k_{m-1}+1}(q',q)$$

are known by the set equality (9). Since $a_{i,i+1}(m-1,m-1)$ are invertible, $t_{i,i+j-k_{m-1}}(m-1,q)$ can be presented by $t_{i-1,i+j-k_{m-1}-1}(m-1,q)$. A straightforword matrix multiplication shows that the successive substitution of the expression of $t_{i,i+j-k_{m-1}}$ by $t_{i-1,i+j-k_{m-1}-1}$ into the equ $k_{m-1}k_{m-1}+i(m-1,m)$ yields

$$T'_{m-1}B^{k_{m-1}}=S_{m-1} \tag{12}$$

where

$$T'_{m-1}=(t_{1j}(m-1,q)),\ S_{m-1}=(S''_{1j}(m-1,q))$$

is known and $S''_{1j}(m-1,q)=0$ for $j\leq k_{m-1}$(because the entries $\beta_{ij}(p,q)$ of $B^{k_{m-1}}$ are zero for $j\leq k_{m-1}$). Since $B^{k_{m-1}}$ is injective on $(\ker B^{k_{m-1}})^{\perp}$, $\dim R(B^{k_{m-1}})=\dim(\ker B^{k_{m-1}})^{\perp}$. Since $R(B^{k_{m-1}})$ is closed, T'_{m-1} is determined on $R(B^{k_{m-1}})$ by (12). Let T_{m-1} be an extension of T'_{m-1} on the subspaoe $\sum_{q=m}^{k_m}\sum^{k_{m-1}} k_i(q)+\sum_{i=1}^{k_m} k_i(m)$ and put

$$t_{1j}(m-1,q)=T_{m-1}\,k_j(q).$$

Then define the other entries $t_{ij}(m-1, q)$ arbitrarily and employ the method used in (3) to obtain the entries $\{t_{ij}(m-1, q), 1\le q\le m\}$.

5) To determine the entries $t_{ij}(p, q)$, $1\le p\le m-2$, we can use closed ness of $R(B^{k_p})$. The difference from (4) is that we should consider more equ′s $k_p \;\; k_p+i(p, q)$, $1\le i\le k_q-k_p$, $p<q\le m$ instead of equ′s (10).

Finally put $T=(t_{ij})$ which is a solution of (1).

Now we are in the position to prove the result.

Theorem 2. Let H, K be complex Hilbert spaces and $A\in B(\mathrm{H})$, $B\in B(\mathrm{K})$ be nilpotent operators of order n, the equation (1) has solutions for $S\in \overline{R(D_{AB})}$ if at least one of the following conditions is satisfied:

i) $I(A)=\varnothing$; ii) $I(B)=\varnothing$; iii) $C(A)=C(B)=\{1,2,\cdots, n-1\}$. iv) $I(A)\subset C(B)$; v) $C(A)\supset I(B)$.

Proof. For case i) we apply proposition 1. For case iii) we apply Lemma 4.2 of [1] For case iv) we apply Proposition 2. For case ii) and v) we consider $D_{B^*A^*}$, then apply Theorem 3 of [2], Lemma 2.2 of [1] and Proposition 1 and 2.

In order to illustrate the methed used above we give two examples here.

Example 1. Assume $A^3=B^3=0$, $I(A)=C(B)=\{2\}$. Then set

$$H_5'=\ker A^3\ominus\ker A^2,\quad H_4'=A_2H_5',\quad H_3'=A_1H_4',$$

$$H_1'=\ker A\ominus H_3',\quad H_2'=\{x\in(\ker A^2\ominus\ker A),\ A_1x\in H_1'\}.$$

Similarly define $K_i's$. Thus

$$A=\left[\begin{array}{cc|ccc} 0 & a_2 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a_4 & * \\ 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\qquad B=\left[\begin{array}{cc|ccc} 0 & b_2 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_4 & * \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\qquad B=\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & b_4b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where a_2, a_4, a_5, b_4、b_5 are invertible. $R(b_4b_5)=R(B^2)$ is closed. The equation (1) can be written as

$$\begin{bmatrix} a_2t_{21}+*t_{51} & a_2t_{22}+*t_{52}-t_{11}b_2 & a_2t_{23}+*t_{53} & a_2t_{24}+*t_{54}-t_{13}b_4 & a_2t_{25}+*t_{55}-t_{11}*-t_{13}*-t_{14}b_5 \\ 0 & -t_{21}b_2 & 0 & -t_{23}b_4 & -t_{21}*-t_{23}*-t_{24}b_5 \\ a_4t_{41}+*t_{51} & a_4t_{42}+*t_{52}-t_{31}b_2 & a_4t_{43}+*t_{53} & a_3t_{44}+*t_{54}-t_{33}b_4 & a_4t_{45}+*t_{55}-t_{31}*-t_{33}*-t_{34}b_5 \\ a_5t_{51} & a_5t_{52}-t_{41}b_2 & a_5t_{53} & a_5t_{54}-t_{43}b_4 & a_5t_{55}-t_{41}*-t_{43}*-t_{44}b_5 \\ 0 & -t_{51}b_2 & 0 & -t_{53}b_4 & -t_{51}*-t_{53}*-t_{54}b_5 \end{bmatrix}=(S_{ij})$$

By Remark 2 we need not consider the equ′s (5, 1)−(5, 5) and (2, 1) −(2, 4).

Solution i) Since a_2, a_4 and a_5 are invertible we can solve the equ′s corresponding to the columns 1, 3 to obtain t_{51}, t_{41}, t_{21}, t_{53}, t_{43}, and t_{23}. (Note that these entries automatically satisfy the equ's (2, 2) (2, 4), (5, 2), (5, 4)).

ii) Solve the equ's (4, 2) and (4, 4) to obtain t_{52}, t_{54}. (Note that t_{51}, t_{53} and t_{54} automatically satisfy the equ (5, 5)).

iii) Define t_{3j} arbitrarily for $1 \leq j \leq 5$ and solve the equ's (3, 2), (3, 3), (4, 5), (3, 5) in turn to obtain t_{42}, t_{44}, t_{55}, and t_{45}.

iv) Solve the equ's (1, 4), and (2, 5), i.e.,

$$a_2 t_{24} - t_{13} b_4 = S'_{14} \tag{13}$$

$$- t_{24} b_5 = S'_{25} \tag{14}$$

where $S'_{14} = S_{14} - * t_{54}$, $S'_{25} = S_{25} + t_{21} * + t_{23} *$. From (13) we have

$$t_{24} = a_2^{-1} (S'_{14} + t_{13} b_4) \tag{15}$$

The substitution of (15) into (14) yields $t_{13} B^2 = -(a_2 S'_{25} + S'_{14} b_5)$. Since $R(B^2)$ is closed and equal to K'_5 we can define

$$t_{13} B^2 y = -(a_2 S'_{25} + S'_{14} b_5) y, \quad \forall y \in K'_5 .$$

Then determine t_{24} by (15).

v) Define t_{11}, t_{12}, t_{14}, t_{15} arbitrarily and determine t_{22} by equ(1, 2), t_{25} by (1, 5).

Example 2. Assume $A^3 = B^3 = 0$, $I(A) = C(B) = \{1\}$. Then set

$$H'_1 = H_1 \ominus R(A_1), \ H'_2 = R(A_1), \ H'_3 = H_2, \ H'_4 = H_3 .$$

Similarily define K'_is. Thus

$$A = \left[\begin{array}{c|ccc} 0 & 0 & 0 & * \\ \hline 0 & 0 & a_3 & * \\ 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{c|ccc} 0 & 0 & 0 & * \\ \hline 0 & 0 & b_3 & * \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

where a_3, a_4 and b_3 are invertible, $R(B)$ is closed. The equ (1) can be written as

$$\begin{bmatrix} * t_{41} & * t_{42} & * t_{43} - t_{12} b_3 & * t_{44} - t_{11} * - t_{12} * - t_{13} b_4 \\ a_3 t_{31} + * t_{41} & a_3 t_{32} + * t_{42} & a_3 t_{33} + * t_{43} - t_{22} b_3 & a_3 t_{34} + * t_{44} - t_{21} * - t_{22} * - t_{23} b_4 \\ a_4 t_{41} & a_4 t_{42} & a_4 t_{43} - t_{32} b_3 & a_4 t_{44} - t_{31} * - t_{32} * - t_{33} b_4 \\ 0 & 0 & - t_{42} b_3 & - t_{41} * - t_{42} * - t_{43} b_4 \end{bmatrix} = (S_{ij})$$

Similarly to the example 2. we can determine the entries $\{t_{ij}, \ i \geq 2\}$.

Since B is injective on $K'_3 \oplus K'_4$, $\dim(K'_3 \oplus K'_4) = \dim R(B_{K'_3 \oplus K'_4})$. Since $R(B_{K'_3 \oplus K'_4})$ is closed, we can define an operator T'_1 on $R(B_{K'_3 \oplus K'_4}) \subset K'_1 \oplus K'_2 \oplus K'_3$ by the equation

$$T'_1 B y = (0 \quad 0 \quad S_{13} - * t_{43} \quad S_{14} - * t_{44}) y, \quad \forall y \in K'_3 \oplus K'_4 .$$

Let T_1 be an extension of T'_1 on $K'_1 \oplus K'_2 \oplus K'_3$ and then put

$$t_{11} = T_{1k_1'},\quad t_{12} = T_{1k_2'},\quad t_{13} = T_{1\tilde{k}_3}.$$

Finally define t_{14} arbitrarily and let $T = (t_{ij})$ which is a solution of (1)

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