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Conditions for the uniqueness of statistic's distribution in the class of spherical distridutions

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This paper gives the necessary and sufficient conditions, which are very simple for checking the uniqueness of the distribution of a statistic in those classes of the matrix spherical distributions and improve the results of Kariya [1].

Many properties of the matrix spherical distribution are similar to those of normal distribution, therefore to find the conditions, under which the distribut tion of a statistic remains the same in the class of matrix spherical distribution tions, is a valuable problem in the multivariate statistical analysis, and it is also important for deriving the null distributions of some testing statistics and for researching robustness. Some authors discussed this problem, Kariya [1] obtained the necessary and susufficient condions, which were very simple for checcking the uniqueness of the distribution of a statistic in those classes of matrix spherical bistributions. His results are very nice, but his main theorem is not completely correct. In this paper, we also give such necessary and sufficient conditions. which improve the results of [1].

Let $\mathcal{O}(n)$ be the set of $n\times n$ orthogonal matrices, $\mathscr{S}(p)$ be the set of $p\times$ postiive definite matrices, $Gl(p)$ be the set of $p\times p$ nonsingular matrices, and $GU(p)$ (or $GT(p)$) be the set of $p\times p$ upper (or lower) triangular matrices with positive diagonal elements. Let $\mathscr{L}(X)$ be the distribution of X.

Definition Let $X=(x_1, \cdots, x_p)$ be an $n\times p$ $(n\geq p)$ random matrix.

(i) if $\mathscr{L}(\Gamma X)=\mathscr{L}(X)$ for all $\Gamma\in\mathcal{O}(n)$, the distribution of X is called to the left $\mathcal{O}(n)$-invariant and the set of all distributions with this property is denoted by $\mathscr{F}_3$. $\mathscr{F}_3$ is called the class of left $\mathcal{O}(n)$-invariant distributions al also.

(ii) let $\mathscr{F}_2$ be the set of distributions with the following property: $\mathscr{L}((\Gamma_1 x_1, \cdots, \Gamma_p x_p))=\mathscr{L}((x_1, \cdots, x_p))$ holds for all $\Gamma_i\in\mathcal{O}(n)$, $i=1,\cdots, p$.

(iii) let $\mathscr{F}_1$ be the set of all distributions with the following property: $\mathscr{L}(\Gamma \mathrm{Vec}X)=\mathscr{L}(\mathrm{Vec}X)$ holds for all $\Gamma\in\mathcal{O}(np)$, where $\mathrm{Vec}X=(x_{(1)}, \cdots, x_{(n)})'$, $x_{(1)}, \cdots, x_{(n)})'$, $x_{(j)}$ is the jth row of X, $j=1,\cdots, n$.

$\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ are all called the classes of matrix spherical distributions.

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Obviously, $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3$. The properties of these distributions can be found in [3] and [4]. Let $\mathscr{F}_i^*$ be the subclass consisted of all absolutely continuous distributions of $\mathscr{F}_i$, $i=1,2,3$, and $\mathscr{N}=\{N(0, I_n, \Sigma): \Sigma \in \mathscr{S}(p)\}$ be the subclass of normal distributions with zero mean matrix. Obviously, $\mathscr{N} \subset \mathscr{F}_3^*$ and $\mathscr{F}_1^* \subset \mathscr{F}_2^* \subset \mathscr{F}_3^*$.

For $F \in \mathscr{F}_i$, let $\mathscr{F}_i(F)$ be the following subclass of $\mathscr{F}_i$:

$$\mathscr{F}_i(F)=\{F^*: dF^* \ll dF,\ F^* \in \mathscr{F}_i\},\ i=1,2,3.$$

Lemma 1 Let $X=(x_1,\cdots,x_p)$ be $n \times p$ $(n \geqslant p)$ random matrix, $\|X\|^2 = \operatorname{tr} X'X$, and $\|x_j\|^2 = x_j' x_j$, $j=1,\cdots,p$.

(i) If $\mathscr{L}(X) \in \mathscr{F}_1$ and $P(\|X\|=0)=0$, then X can be rewritten as $X=\frac{X}{\|X\|} \cdot \|X\| \triangleq U_1 R$, where $R=\|X\| > 0$ and $U_1=\frac{X}{\|X\|}$ are independent, $\mathscr{L}(\mathrm{Vec}U_1) = \mathscr{L}(u^{(np)})$.

(ii) If $\mathscr{L}(X) \in \mathscr{F}_2$ and $P(\|x_j\|=0)=0$, $j=1,\cdots,p$, then X can be expressed as $X=\left(\frac{x_1}{\|x_1\|},\cdots,\frac{x_p}{\|x_p\|}\right) \cdot \operatorname{diag}(\|x_1\|,\cdots,\|x_p\|) \triangleq U_2 R$, where $R=\operatorname{diag}(R_1,\cdots,R_p)$ and $U_2=\left(\frac{x_1}{\|x_1\|},\cdots,\frac{x_p}{\|x_p\|}\right)=(u_1,\cdots,u_p)$ are independent, $R_j=\|x_j\| \geqslant 0$, $j=1,\cdots,p$, $u_1,\cdots,u_p$ are independent and identically distributed, $\mathscr{L}(u_1)=\mathscr{L}(u^{(n)})$.

(iii) If $\mathscr{L}(X) \in \mathscr{F}_3$ and $P(|X'X|=0)=0$, then X can be expressed as $X=U_3 A$, where $A \in \mathscr{S}(p)$ is independent of U_3, $\mathscr{L}(A)=\mathscr{L}((X'X)^{1/2})$, $\mathscr{L}(U_3) \in \mathscr{F}_3$, $U_3' U_3 = I_p$, whene $\mathscr{L}(u^{(n)})$ is the uniform distribution on the unit sphere in $R^{(n)}$, $P(\|u^{(n)}\|=1)=1$, $\mathscr{L}(U_3)$ is the unique uniform distribution on the Stiefel manifold. (see [2])

Lemma 2 Let μ, ν be the two σ—finite measures defined on the measurable space $(\Omega\ \mathscr{B})$, $\nu \ll \mu$, W and X be the random vectors defined on $(\Omega\ \mathscr{B})$, $\mathscr{B}(X) \subseteq \mathscr{B}(W)$ ($\mathscr{B}(X)$ is the Borel σ-field generated by X), $\frac{d\nu^W}{d\mu^W}$ and $\frac{d\nu^X}{d\mu^X}$ be the derivatives of the ν distributions with regard to μ distributions of W and X respectively, E_ν and E_μ be the expectation operators un er the measures ν and μ respectively. Then for every bounded Borel measurable function f,

$$E_\nu(f(W) \mid X) = \left(\frac{d\nu^X}{d\mu^X}\right)^{-1} E_\mu\left(f(W)\frac{d\nu^W}{d\mu^W} \,\Big|\, X\right), \text{ a.e. } \nu \tag{1}$$

Proof. For every Borel set B, $A=X^{-1}(B) \in \mathscr{B}(X) \subseteq \mathscr{B}(W)$,

$$\int_A E_\nu(f(W) \mid X)\frac{d\nu^X}{d\mu^X}(X)\, d\mu = \int_B E_\nu(f(W) \mid X)\, d\nu^X$$

$$=\int_A f(\mathrm{W})\,\mathrm{d}\nu=\int_A f(\mathrm{W})\ \frac{\mathrm{d}\nu^{\mathrm{W}}}{\mathrm{d}\mu^{\mathrm{W}}}\ (\mathrm{W})\,\mathrm{d}\mu=\int_A E_\mu(f(\mathrm{W})\frac{\mathrm{d}\nu^{\mathrm{W}}}{\mathrm{d}\mu^{\mathrm{W}}}\mid \mathrm{X})\,\mathrm{d}\mu.$$

From this we obtain

$$E_\nu(f(\mathrm{W})\mid\mathrm{X})\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}}=E_\mu(f(\mathrm{W})\ \frac{\mathrm{d}\nu^{\mathrm{W}}}{\mathrm{d}\mu^{\mathrm{W}}}\mid\mathrm{X}),\qquad \text{a.e.}\ \mu\ (\text{a.e.}\ \nu).$$

Since

$$\nu(\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}}=0)=0,$$

we get

$$E_\nu(f(\mathrm{W})\mid\mathrm{X})=(\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}})^{-1}E_\mu(f(\mathrm{W})\frac{\mathrm{d}\nu^{\mathrm{W}}}{\mathrm{d}\mu^{\mathrm{W}}}\mid\mathrm{X}),\qquad \text{a.e.}\ \nu.$$

Lemma 3. Let ν^{X}, ν^{Y}, ν^{Z} and $\mu^{\mathrm{X}},\mu^{\mathrm{Y}},\mu^{\mathrm{Z}}$ be the distributions of X, Y, Z under ν and μ respectively. Suppose

(i) Z = YX, X is a measurable function of Z, Y is independent of X for ν and μ,

(ii) $\nu^{\mathrm{Y}}=\mu^{\mathrm{Y}}$ and $\nu^{\mathrm{Z}}\ll\mu^{\mathrm{Z}}$,

(iii) $t(\mathrm{Z})$ is a measurable function of Z, and $t(\mathrm{Z})$ is independent of X under μ.

Then (i) $t(\mathrm{Z})$ is still independent of X under ν, and (ii) the distrbution of t(Z) under μ is same as that under ν.

Proof. Since Y is independent of X, X is a measurable function of Z, $\nu^{\mathrm{Z}}\ll\mu^{\mathrm{Z}}$ and $\nu^{\mathrm{Y}}=\mu^{\mathrm{Y}}$, we can obtain $\nu^{\mathrm{X}}\ll\mu^{\mathrm{X}}$.

Let W = (Y, X), then

$$\nu^{\mathrm{W}}=\nu^{\mathrm{Y}}\times\nu^{\mathrm{X}}=\mu^{\mathrm{Y}}\times\nu^{\mathrm{X}},\quad \mu^{\mathrm{W}}=\mu^{\mathrm{Y}}\times\mu^{\mathrm{X}},\quad \nu^{\mathrm{W}}\ll\mu^{\mathrm{W}},$$

and

$$\frac{\mathrm{d}\nu^{\mathrm{W}}}{\mathrm{d}\mu^{\mathrm{W}}}(\mathrm{W})=\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}}(\mathrm{X}(\mathrm{W})).$$

Let

$$\mathrm{Z}=\mathrm{YX}=f(\mathrm{W}),$$

$$\mathrm{T}=t(\mathrm{Z})=t(f(\mathrm{W}))=g(\mathrm{W}).$$

Then $t(\mathrm{Z})$ is indepdenent of X under μ by the condition (iii). Therefore, for any bounded Borel measurable function h,

$$E_\mu(h(\mathrm{T})\mid\mathrm{X})=E_\mu(h(\mathrm{T})).$$

Using lemma 2, we get

$$E_\nu(h(T)\mid\mathrm{X})=(\frac{\mathrm{d}\mu^{\mathrm{X}}}{\mathrm{d}\nu^{\mathrm{X}}})^{-1}E_\mu(h(g(\mathrm{W}))\frac{\mathrm{d}\nu^{\mathrm{W}}}{\mathrm{d}\mu^{\mathrm{W}}}(\mathrm{W})\mid\mathrm{X})=(\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}})^{-1}E_\mu(h(g(\mathrm{W}))$$

$$\cdot\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}}(\mathrm{X})\mid\mathrm{X})=(\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}})^{-1}\frac{\mathrm{d}\nu^{\mathrm{X}}}{\mathrm{d}\mu^{\mathrm{X}}}E_\mu(h(\mathrm{T})\mid\mathrm{X})=E_\mu(h(\mathrm{T})\mid\mathrm{X})$$

$= E_\mu h(T)$, a. e. ν.

Hence, $T = t(Z)$ is independent of X under ν and the distribution of $t(Z)$ under ν is the same as that under μ.

At first, we find the conditions under which the distribution of $t(X)$ remains the same in the class $\mathscr{F}_3$ or $\mathscr{F}_3^*$.

Theorem 1. Suppose X is an $n \times p$ random matrix and $t(X)$ is a statistic. If when $\mathscr{L}(X) = F \in \mathscr{F}_3$ and $P(|X'X| = 0) = 0$, $\mathscr{L}(t(X)) = \mathscr{L}(t(U_3))$ holds, then $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X) \in \mathscr{F}_3(F)$. i.e.

$$\mathscr{L}(t(X)) = \mathscr{L}(t(U_3)), \text{for all } \mathscr{L}(X) \in \mathscr{F}_3(F), \qquad (2)$$

where U_3 is defined at lemma 1.

Proof Suppose $\mathscr{L}(X) = F \in \mathscr{F}_3$ and $P(|X'X| = 0) = 0$. By lemma 1, $X = U_3 A$, where $A \in \mathscr{S}(p)$, A is independent of U_3, and $\mathscr{L}(U_3)$ is the uniform distribution on the Stiefel manifold. Since $\mathscr{L}(t(X)) = \mathscr{L}(t(U_3))$ for $\mathscr{L}(X) = F$, $t(X)$ is independent of A under $\mathscr{L}(X) = F$. For any $\mathscr{L}(X) = F^* \in \mathscr{F}_3(F)$, $dF^* \ll dF$. Using lemma 3, we obtain that $t(X)$ is still independent of A under F^* and that the distribution of $t(X)$ under F is the same as that under F^*. Because of the arbitrariness of F^*, we obtain

$$\mathscr{L}(t(X)) = \mathscr{L}(t(U_3)) \text{ for all } \mathscr{L}(X) \in \mathscr{F}_3(F).$$

Theorem 2 $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X) \in \mathscr{F}_3^*$ iff $\mathscr{L}(t(X)) = \mathscr{L}(t(U_3))$ when $\mathscr{L}(X)$ is the normal distribution $(0, I_n, I_p)$.

Proof (Sufficiency) Let F is the distribution corresponding to $N(0, I_n, I_p)$. Since $dF \ll d\tilde{F}$ for all $F \in \mathscr{F}_3^*$, $\mathscr{F}_3^* = \mathscr{F}_3(\tilde{F})$. The sufficiency is proved by using theorem 1.

(Necessity) Because of $\mathscr{N} \in \mathscr{F}_3^*$ and assumption, the distribution of $t(X)$ remains the same in the class $\mathscr{N}$. Since when $\mathscr{L}(X) \in \mathscr{N}$, $\mathscr{L}(XC) \in \mathscr{N}$ for all $C \in Gl(p)$,

$$\mathscr{L}(t(XC)) = \mathscr{L}(t(X)) \text{ for all } C \in \quad (p) \text{ and } \mathscr{L}(X) \in \mathscr{N}.$$

By lemma 1 (iii), $X = U_3 A$, U_3 and A are independent, $\mathscr{L}(U_3)$ is the uniform distribution on the Stiefel manifold, and $\mathscr{L}(A^2) = \mathscr{L}(X'X)$. Substituting $U_3 A$ for X in the privious expression, we get

$$\mathscr{L}(t(U_3 AC)) = \mathscr{L}(t(U_3 A)) \text{ for all } C \in Gl\ (p) \text{ and all } \mathscr{L}(X) \in \mathscr{N}.$$

Obviously $\{\mathscr{L}(A^2) = \mathscr{L}(X'X) : \mathscr{L}(X) \in \mathscr{N}\} = \{W_p(n, \Sigma) : \Sigma \in \mathscr{S}(p)\}$, where $W_p(\cdot, \cdot)$ is the Wishart distributioon. By the properties of exponential family of distributions, A^2 is a complete statistic for $\{W_p(n, \Sigma) : \Sigma \in \mathscr{S}(p)\}$. Since there is $1:1$ transformation between A^2 and A, A is a complete statistic also.

For any bounded Borel function h, when $\mathscr{L}(X)$ is any $N(0, I_n, \Sigma)$,

$$E_\Sigma h(t(U_3AC)) = E_\Sigma E(h(U_3AC)) \mid A) = E_\Sigma h(t(U_3A)) = E_\Sigma E(h(t(U_3A)) \mid A)$$

by the completeness of A, we obtain

$$E(h(t(U_3AC)) \mid A) = E(h(t(U_3A)) \mid A) \quad \text{a. e. A for any } C \in Gl(p).$$

Taking $C = A^{-1}$ in the previous expression, we obtain

$$E(h(t(U_3A)) \mid A) = Eh(t(U_3)), \quad \text{a. e. A.}$$

Taking the expectation with regard to A yields

$$Eh(t(X)) = Eh(t(U_3)) \text{ for all } \mathcal{L}(X) \in \mathcal{N}$$

By the arbitrariness of h,

$$\mathcal{L}(t(X)) = \mathcal{L}(t(U_3)) \text{ for all } \mathcal{L}(X) \in \mathcal{N}.$$

In particularly, taking $\mathcal{L}(X)$ is the distribution of $N(0, I_n, I_p)$ the conclusion of this theorem is obtained.

Below, we give an example, which shows that when the conditions of theorem 2 hold, $\mathcal{L}(t(X))$ remains the same, but when $\mathcal{L}(X) \in \mathcal{F}_3$ but $\mathcal{L}(X) \notin \mathcal{F}_3^*$, $\mathcal{L}(t(X))$ may be different.

Example 1 Suppose

$$n = 2p, \quad \underset{2p \times p}{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{matrix} p \\ p, \end{matrix}$$

$$t(X) = \begin{cases} \dfrac{|x_1' x_1|}{|X'X|}, & \text{if } |X'X| \neq 2 \\ 5, & \text{if } |X'X| = 2 \end{cases}$$

When $\mathcal{L}(X)$ is any normal distriribution which belongs to the class $\mathcal{N}$, because $P(|X'X| = 2) = 0$ and $P(t(X) = t(U_3)) = 1$, we obtain

$$\mathcal{L}(t(X)) = \mathcal{L}(t(U_3)) \text{ for all } \mathcal{L}(X) \in \mathcal{N}.$$

Therefore $\mathcal{L}(t(X))$ remains the same for all $\mathcal{L}(X) \in \mathcal{F}_3^*$ y using theorem 2. Let E_1 be the distribution function corresponding to $N(0, I_n, I_p)$, F_2 be the distribution function corresponding to $P(|X'X| = 2) = 1$, and $F = 0.5F_1 + 0.5F_2$. It is obvious that $F \in \mathcal{F}_3$, $F \notin \mathcal{F}_3^*$ and

$$P_F(t(X) = 5) = 0.5, \quad P_F(t(U_3) = 5) = 0,$$

$$P_F(t(X) = 5) \neq P_F(t(U_3) = 5).$$

This inequality shows that when $\mathcal{L}(X) = F$, $\mathcal{L}(t(X))$ is different from $\mathcal{L}(t(U_3))$, i.e. $\mathcal{L}(t(X))$ is not unique in the class $\mathcal{F}_3$.

Example 1 shows that the main theorems of the paper [1] are not completely correct. There are some mistakes in the proof of the "if" part of the theorem 1 of [1].

Corollary 1 $\mathcal{L}(t(X))$ remains the same for all $\mathcal{L}(X) \in \mathcal{F}_3^*$ iff when $\mathcal{L}(X)$ is the normal distribution $N(0, I_n, I_p)$,

$$\mathcal{L}(t(XA)) = \mathcal{L}(t(X)), \text{ for all } A \in \mathcal{G}(p).$$

Proof The necessity is obvious. When $\mathcal{L}(X)$ is the normal distribution

$N(0, I_p)$, $\mathscr{L}(XA)$ is the normal distribution $N(0, I_n, A'A) \in \mathscr{N}$. Since $\mathscr{L}(t(X))$ remains the same in $\mathscr{F}_3^*$, $\mathscr{L}(t(XA)) = \mathscr{L}(t(X))$.

Now we prove the sufficiency. Obviously, $\mathscr{N} = \{\mathscr{L}(XA): \mathscr{L}(X)$ is the normal distribution $N(0, I_n, I_p)$, $A \in \mathscr{S}(p)\}$. Therefore the condition (3) implies that $\mathscr{L}(t(X))$ remains the same in $\mathscr{N}$. By the proof of the "only if" part of the theorem 2, we obtain

$$\mathscr{L}(t(X)) = \mathscr{L}(t(U_3)) \text{ for } \mathscr{L}(X) \text{ is the normal distribution } N(0, I_n, I_p).$$

Hence the conclusion of corollary 1 is correct.

Corollary 2 $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X) \in \mathscr{F}_3^*$ iff when $\mathscr{L}(X)$ is the normal distribution $N(0, I_n, I_p)$,

$$\mathscr{L}(t(XA)) = \mathscr{L}(t(X)) \quad \text{for all } A \in GU(p) \quad (\text{or } GT(p)). \qquad (4)$$

Since when $\mathscr{L}(X) \in \mathscr{F}_3^*$, X can be expressed as $X = U_3 A$, where $A \in GU(p)$ (or $GT(p)$), A is independent of U_3, and $\mathscr{L}(U_3)$ is the uniform distribution on the Stiefel manifold, the corollary 2 can be proved similarly as that of corollary 1.

Corollary 3 If $t(X)$ satisfies the condition

$$t(X) = t(U_3) \qquad (5)$$

then $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X) \in \mathscr{F}_3$.

Corollary 4 If $t(X)$ satisfies the condition

$$t(XA) = t(X) \text{ for all } A \in \mathscr{S}(p) \quad (\text{or } GU(p) \text{ or } GT(p)) \qquad (6)$$

then $\mathscr{L}(t(X))$ remains the same in $\mathscr{F}_3$.

The following example shows that the condition (5) or (6) is sufficient, but not necessary.

Example 2 Suppose $\underset{2\times 1}{X} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$, $\mathscr{L}(X) \in \mathscr{F}_3$ and $P(X=0) = 0$.

Let

$$t_1(X) = \frac{x_1}{\|X\|} g(\|X\|), \quad g(a) = \begin{cases} -1, & \text{if } a \neq 1, \\ 1 & \text{if } a = 1. \end{cases}$$

By lemma 1, X can be expressed as $X = Ua$, $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where U is independent of a, and $\mathscr{L}(U)$ is the uniform distribution on the unit circle. By substituting Ua for X in $t_1(X)$, we obtain

$$t_1(X) = \begin{cases} u_1 & \text{if } \|X\| = 1 \\ -u_1 & \text{if } \|X\| \neq 1 \end{cases} \quad \text{and} \quad t_1(U) = u_1.$$

Since $\mathscr{L}(u_1)$ is symmetric about $u_1 = 0$ on the interval $(-1, 1)$, $\mathscr{L}(u_1) = \mathscr{L}(-u_1)$. Hence

$$\mathscr{L}(t_1(X)) = \mathscr{L}(t_1(U)).$$

However, when $\mathscr{L}(X) \in \mathscr{F}_3$ and $P(\|X\|=1)<1$, $P(t_1(X)=t_1(U))<1$ holds. This shows that $t_1(X)=t_1(U)$ a.e. is not true, but $\mathscr{L}(t_1(X))$ is the same as $\mathscr{L}(t_1(U))$.

Next we give the conditions under which $\mathscr{L}(t(X))$ remains the same in $\mathscr{F}_2$ or $\mathscr{F}_2^*$.

Theorem 3 Suppose X is a $n\times p$ random matrix and $t(X)$ is a statistic. If when $\mathscr{L}(X)=F\in\mathscr{F}_2$ and $P(\|x_j\|=0)=0$, $j=1, \cdots p$,

$$\mathscr{L}(t(X))=\mathscr{L}\left(t\left(\frac{x_1}{\|x_1\|}, \cdots, \frac{x_p}{\|x_p\|}\right)\right) \tag{7}$$

holds, then $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X)\in\mathscr{F}_2(F)$.

(The proof is similar to that of theorem 1).

Theorem 4 $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X)\in\mathscr{F}_2^*$.iff

$$\mathscr{L}(t(X))=\mathscr{L}\left(t\left(\frac{x_1}{\|x_1\|}, \cdots, \frac{x_p}{\|x_p\|}\right)\right) \tag{8}$$

holds when $\mathscr{L}(X)$ is the normal distribution $N(0, I_n, I_p)$.

Proof The sufficiency is proved by theorem 3. The proof of "only if" part is similar to that of theorem 2, we only need to note that $\{\mathscr{L}(\|x_1\|, \cdots, \|x_p\|): \mathscr{L}(X)$ is the normal distribution $N(0, I_n, \Sigma)$, where $\Sigma=\operatorname{diag}(\sigma_1^2, \cdots, \sigma_p^2)$, $\sigma_i^2>0$, $i=1, \cdots, p.\}=\{\mathscr{L}(R_1, \cdots, R_p): R_p$ are independent, $R_j^2/\sigma_j^2\sim\chi^2(n)$, $\sigma_j^2>0$, $j=1, \cdots, p\}$ and $(\|x_1\|^2, \cdots, \|x_p\|^2)$ is a complete statistic for this family of distributions.

Corollary 5 $\mathscr{L}(t(X))$ remains the same for all $\mathscr{L}(X)\in\mathscr{F}_2^*$ iff when $\mathscr{L}(X)$ is the normal distribution $N(0, I_n, I_p)$,

$$\mathscr{L}(t(XD))=\mathscr{L}(t(X)) \text{ for all } D=\operatorname{diag}(d_1, \cdots, d_p) \text{ with } d_j>0,\ j=1, \cdots, p. \tag{9}$$

Corollary 6. If $t(X)$ satisfiess the condition

$$t(X)=t\left(\left(\frac{x_1}{\|x_1\|}, \cdots, \frac{x_p}{\|x_p\|}\right)\right) \tag{10}$$

or

$$t(XD)=t(X) \text{ for all } D=\operatorname{diag}(d_1, \cdots, d_p) \text{ with } d_j>0,\ j=1, \cdots, p, \tag{11}$$

then $\mathscr{L}(t(X))$ remains the same in $\mathscr{F}_2$ with $P(\|x_j\|=0)=0$, $j=1, \cdots, p$.

Finally, we give the conditions, under which the $\mathscr{L}(t(X))$ remains the same in $\mathscr{F}_1$ or $\mathscr{F}_1^*$.

Theorem 5. Suppose X is an $n\times p$ random matrix nd $t(X)$ is a statistic. If when $\mathscr{L}(x)=F\in\mathscr{F}_1$ and $P(\|X\|=0)=0$,

$$\mathscr{L}(t(X))=\quad(t(X/\|X\|)) \tag{12}$$

holds, then $\mathscr{L}(t(X))$ remains the same in $\mathscr{F}_1(F)$.

Theorem 6 $\varphi(t(X))$ remains the same for all $\varphi(X) \in \mathscr{F}_1^*$ iff when $\varphi(X)$ is the normal distribution $N(0, I_n, I_p)$,

$$\varphi(t(X)) = \varphi(t(X/\|X\|)) \tag{13}$$

holds.

Corollary 7 $\varphi(t(X))$ remains the same for all $\varphi(X) \in \mathscr{F}_1^*$ iff when $\varphi(X)$ is the normal distribution,

$$\varphi(t(aX)) = \varphi(t(X)) \text{ for all } a > 0. \tag{14}$$

Corollary 8 If $t(X)$ satisfies the condition

$$t(X) = t(X/\|X\|) \tag{15}$$

or

$$t(aX) = t(X) \text{ for all } a > 0, \tag{16}$$

then $\varphi(t(X))$ remains the in $\mathscr{F}_1$ with $P(\|X\| = 0) = 0$.

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