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# Matrix Algebra Motivated by Essentially Stochastic Matrices*

*Jin Bai Kim*

(West Virginia University, Morgantown, W. VA.)

**Abstract**

A matrix of order $n$ whose row sums are all equal to 1 is called an essentially stochastic matrix (see Johnsen [4]). We extend this notion as the following. Let F be a field of characteristic 0 or a prime greater than $n$. $M_n(F)$ denotes the set of all $n\times n$ matrices over F. Let $t$ be an element of F. A matrix $A=(a_{ij})$ in $M_n(F)$ is called essentially $t$-stochastic' provided its row sums are each equal to $t$. We denote by $R_n(t)$ the set of all essentially $t$-stochastic matrices over F. We shall mainly study $R_n(0)$ and $R_n(F)=\bigcup_{t\in F} R_n(t)$. Our main references are Johnson [2,4] and Kim [5].

**1. Introduction Raview** Let F be a field of characteristic 0 or a prime greater than $n$. We denote the set of all $n\times n$ matrices over F by $M_n(F)$. For $t\in F$, we denote by $R_n(t)$ the set of all matrices $A=(a_{ij})$ of order $n$ in $M_n(F)$ such that $\sum_{j=1}^{n} a_{ij}=t$. Any $A$ in $R_n(t)$ will be called an essentially $t$-stochastic matrix. We shall study $R_n(F)=\bigcup_{t\in F} R_n(t)$ (which is a multiplicative semigroup). We review papers. Johnsen [4] studied $R_n(1)$ and proved that any matrix $A$ in $R_n(1)$ is a product of at most $q(n)$ elementary matrices in $R_n(1)$, where $q(x)$ is a quadratic polynomial in $x$ when $F\neq GF(2)$. Johnsen also established a similar result for $F=GF(2)$. For $A\in M_n(F)$, $A^T$ denotes the transpose of $A$. Define $L_n(t)=\{A^T: A\in R_n(t)\}$ and $S_n(t)=R_n(t)\cap L_n(t)$. Any matrix in $S_n(1)$ is called an essentially doubly stochastic matrix. In [2], Johnsen proved that $S_n(1)$ has an algebra structure $\{S_n(1), \oplus, \otimes, \times\}$ when the matrix sum $\oplus$ is defined by $A\oplus B=A+B-J_n$, $\otimes$ is defined by $A\otimes B=AB$ (the usual matrix product) and a scalar multiplication $a\times A$ is defined by $a\times A=aA+(1-a)J_n$, for $A, B\in S_n(1)$, for $a\in F$ and where $J_n$ is the matrix with every entry of $J_n$ is equal to $1/n$.

Johnsen [2] proved that $S_n(1)$ is algebra-isomorphic to $M_{n-1}(F)$.

Let $M_1=\{A=(a_{ij})\in M_n(F): a_{21}=a_{31}=\cdots=a_{n1}=0\}$. We shall show that $R_n(F)$

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is algebra-isomorphic onto $M_1$. We shall generalize the following well known result.

**Theorem A**. (See Theorem 5.4[8, p.58]). Let $A$ be a doubly stochastic matrix of order $n$. Then $A = c_1P_1 + c_2P_2 + \cdots + c_kP_k$, where $P_i$ are permutation matrices and the $c_i$ are positive reals such that $c_1 + c_2 + \cdots c_k = 1$. We shall compute the number of idempotents in the set $S_n(t)$.

2. $R_n(F)$. We begin with the following lemma.

**Lemma 1** Under the usual matrix operations, $R_n(0)$ and $R_n(F)$ are algebras over F.

**Proof** Let $s, t \in F$. Let $A \in \mathbf{R}_n(s)$ and $B \in \mathbf{R}_n(t)$. Then we can see that $A + B \in \mathbf{R}_n(s+t)$, $AB \in \mathbf{R}_n(st)$ and $tA \in \mathbf{R}_n(ts)$. This proves the lemma.

We define $\mathbf{M}_1 = \{A = (a_{ij}) \in M_n(F) : a_{i1} = 0$ for $i = 2, 3, \cdots, n\}$ and $M_2 = \{A = (a_{ij}) \in M_n(F) : a_{i1} = 0$ for all $i\}$. We shall prove the following theorem.

**Theorem 1** (1) $R_n(F)$ and $\mathbf{M}_1$ are algebra-isomorphic. (2) $\mathbf{R}_n(0)$ and $\mathbf{M}_2$ are algebra-isomorphic. (3) $R_n(0)$ is an ideal of $R_n(F)$ and the quotien algebra algebra $R_n(F)/R_n(0)$ is isomorphic onto F.

**Proof** (1) We define a non-singular matrix $X = (x_{ij}) \in M_n(F)$ by

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 1 \end{bmatrix}$$

($x_{ii} = 1 = x_{1i}$, $x_{i+1,i} = -1$ and $x_{ij} = 0$ for all other $i$ and $j$). Let $X^{-1} = Y = (y_{ij})$. Then we can see that:

$$Y = \begin{bmatrix} \frac{1}{n} & \frac{-(n-1)}{n} & \frac{-(n-2)}{n} & \frac{-(n-3)}{n} & \frac{-(n-4)}{n} & \cdots & \frac{-3}{n} & \frac{-2}{n} & \frac{-1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{-(n-2)}{n} & \frac{-(n-3)}{n} & \frac{-(n-4)}{n} & \cdots & \frac{-3}{n} & \frac{-2}{n} & \frac{-1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{2}{n} & \frac{-(n-3)}{n} & \frac{-(n-4)}{n} & \cdots & \frac{-3}{n} & \frac{-2}{n} & \frac{-1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{2}{n} & \frac{3}{n} & \frac{-(n-4)}{n} & \cdots & \frac{-3}{n} & \frac{-2}{n} & \frac{-1}{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n} & \frac{2}{n} & \frac{3}{n} & \frac{4}{n} & \cdots & \frac{n-3}{n} & \frac{n-2}{n} & \frac{-1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{2}{n} & \frac{3}{n} & \frac{4}{n} & \cdots & \frac{n-3}{n} & \frac{n-2}{n} & \frac{n-1}{n} \end{bmatrix}.$$

$(y_{i1}=1/n,\ y_{ii}=(i-1)/n(i\neq 1),\ y_{ij}=(j-1)/n(i>j$ and $i\geqslant 2$, and $y_{ij}=-(n-j+1)/n(j>i))$. Letting $XAY=D=(d_{ij}),(A\in R_n(t))$, we see that $d_{i1}=0$ for $i\neq 1$, and hence $D\in M_1$. We define $\phi_X$ by $\phi_X(A)=XAY(A\in R_n(t))$. We see that $\phi_X(AB)=\phi_X(A)\phi_X(B)$, $\phi_X(A+B)=\phi_X(A)+\phi_X(B)$ and $\phi_X(aA)=a\phi_X(A)$ for $A$, $B$ in $R_n(F)$ and $a$ in F. It is clear that $\phi_X$ is one-to-one. We shall show that the mapping $\phi_X$ is onto.

Let $C=(c_{ij})\in M_1$ and $c_{11}=t\neq 0$. Consider $YCX=D=(d_{ij})$. We can compute the following $d_{ij}$:

$$d_{11}=\frac{1}{n}\Big(t-c_{12}+\sum_{k=1}^{n-1}(n-k)c_{k+1,2}\Big),$$

$$d_{1j}=\frac{1}{n}\Big(t+c_{1j}-c_{1,j+1}-\sum_{k=1}^{n-1}(n-k)(c_{k+1,j}-c_{k+1,j+1})\Big),\qquad(2\leqslant j\leqslant n-1),$$

$$d_{1n}=\frac{1}{n}\Big(t+c_{1n}+\sum_{k=1}^{n-1}(n-k)c_{k+1,n}\Big),$$

$$d_{i1}=\frac{1}{n}\Big(t-c_{12}-\sum_{k=2}^{i}(k-1)c_{k2}+\sum_{k=i}^{n-1}(n-k)c_{k+1,2}\Big),\qquad(i\geqslant 2),$$

$$d_{ij}=\frac{1}{n}\Big(t+c_{1j}-c_{1,j+1}+\sum_{k=2}^{i}(k-1)(c_{kj}-c_{k,j+1})-\sum_{k=i}^{n-1}(n-k)(c_{k+1,j}-c_{k+1,j+1})\Big)$$
$$(i\geqslant 2,\ n-1\geqslant j\geqslant 2),$$

$$d_{in}=\frac{1}{n}\Big(t+c_{1n}+\sum_{k=2}^{i}(k-1)c_{kn}-\sum_{k=i}^{n-1}(n-k)c_{k+1,n}\Big),\qquad(i\geqslant 2).$$

Therefore we obtain that $d_{11}+\sum_{j=2}^{n-1}d_{1j}+d_{1n}=t$ and $d_{i1}+\sum_{j=2}^{n-1}d_{ij}+d_{in}=t$, $(i\geqslant 2)$.

Thus $YCX=D\in R_n(t)$, and hence $\phi_X R_n(F)=M_1$. This proves (1).

From the preceding argument it follows that $\phi_X R_n(0)=M_2$ and hence $R_n(0)$ is algebra-isomorphic onto $M_2$. We consider (3). Let $I_n$ be the identity of $M_n(F)$. Then we can show that $R_n(t)=tI_n+R_n(0)$. From this we can obtain that $R_n(s)R_n(t)=(sI_n+R_n(0))(tI_n+R_n(0))=stI_n+R_n(0)$, $R_n(s)+R_n(t)=(s+t)I_n+R_n(0)=R_n(s+t)$ and $aR_n(t)=(at)I_n+R_n(0)=R_n(at)$. (See the proof of Lemma 1). Therefore we proved that $R_n(0)$ is an ideal of $R_n(F)$ and the quotient algebra $R_n(F)/R_n(0)$ is isomorphic onto F. This proves (3) and we proved the theorem 1.

We recall that $L_n(t)=\{A=(a_{ij})\in M_n(F):\sum_{i=1}^{n}a_{ij}=t$ for all $j\}$. Let $L_n(F)=\bigcup_{t\in F}L_n(t)$. Define $N_1=\{A=(a_{ij})\in M_n(F):a_{1j}=0$ for $2\leqslant j\leqslant n\}$ and $N_2=\{A=(a_{ij})\in N_1:a_{11}=0\}$. We have the following.

**Corollary** (1) $L_n(F)$ and $N_1$ are algebra-isomorphic. (2) $L_n(0)$ and $N_2$ are algebra-isomorphic. (3) $L_n(0)$ is an ideal of $L_n(F)$. $L_n(F)/L_n(0)$ is isomorphic to F.

**Proof** Let $X$ be the matrix defined in the proof of Theorem 1. For $A=B^{\mathrm{T}}\in \mathrm{L}_n(t)$, define a mapping $\phi_X$ by $\phi_X(A)=Y^{\mathrm{T}}AX^{\mathrm{T}}=Z^{-1}AZ=(XBX^{-1})^{\mathrm{T}}$, where $Z=X^{\mathrm{T}}$ and $Y=X^{-1}$. From Theorem 1 with this mapping the corollary follows.

**Note** $t$ is an eigenvalue of $A\in \mathrm{R}_n(t)$.

## 3 Some Additional Results

In [5] we proved the following theorem.

**Theorem B** If $A$ is an essentially doubly stochastic matrix ($A\in \mathrm{S}_n(1)$), then $A=c_1P_1+c_2P_2+\cdots+c_kP_k$, where the $P_i$ are permutation matrices and the $c_i$ are constants in F such that $c_1+c_2+\cdots+c_k=1$.

From Theorem B we have the following

**Theorem 2** Let $B\in \mathrm{S}_n(t)$ be an essentially doubly $t$-stochastic matrix. Then $B=\mathrm{d}_1P_1+\mathrm{d}_2P_2+\cdots+\mathrm{d}_kP_k$, where the $P_i$ are permutation matrices and $d_i$ are constants in F such that $d_1+d_2+\cdots+d_k=t$.

**Proof** Let $B\in \mathrm{S}_n(t)$ and $t\neq 0$. Then $\frac{1}{t}B=A$ is a matrix as in Theorem B. Thus the theorem follows from Theorem B with $B=tA$ and $tc_i=d_i$. Let $B\in \mathrm{S}_n(0)$. Consider $C=B+tI_n$, $t\neq 0$. It is clear that $C\in \mathrm{S}_n(t)$ and hence $C$ has an expression of the form $C=\Sigma d_iP_i$. We note that $C=B+tI_n$, $B=C-tI_n$ and the identity $I_n$ is a permutation matrix. This proves the theorem.

We now consider the multiplicative semigroup $\mathrm{S}_n(\mathrm{F})$. Let $P=(p_{ij})$ be a matrix defined by

$$P=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

($p_{nl}=1$, $p_{i,i+1}=1$ for $i=1,2,\cdots,n-1$, and $p_{ij}=0$ for all other $i$ and $j$).

Define $C_n(\mathrm{F})=\{A\in \mathrm{S}_n(\mathrm{F}): A=a_1P^2+a_2P^2+\cdots+a_nP^n\}$, $(P^n=I_n)$.

**Theorem 3** $C_n(\mathrm{F})$ is a maximal commutative subsemigroup of the semigroup $\mathrm{S}_n(\mathrm{F})$.

**Proof** It is clear that $C_n(\mathrm{F})$ is commutative. Suppose that $\mathrm{C}_n(\mathrm{F})$ is not a maximal commutative subsemigroup of $\mathrm{S}_n(\mathrm{F})$. Then there exists a commutative subsemigroup G of $\mathrm{S}_n(\mathrm{F})$ such that $\mathrm{C}_n(\mathrm{F})\subset \mathrm{G}$ and $\mathrm{C}_n(\mathrm{F})\neq \mathrm{G}$. Let $A=(a_{ij})\in \mathrm{G}$ and $A\notin \mathrm{C}_n(\mathrm{F})$. Then we must have that $AP=PA$. From this $A$ takes the form $A=a_{12}P+a_{13}P^2+\cdots+a_{1n}P^{n-1}+a_{nl}P^n$ which is a contrabiction. This proves the theorem.

Let $J_n(t)$ be a matrix such that every entry of $J_n(t)$ is equal to $t/n$.

**Theorem 4** Let $t\in$ F and $t\neq 0$. $\mathrm{S}_n(t)$ has an algebra structure $\{\mathrm{S}_n(t)$, $\oplus$,

$\otimes$, $\times$} under the following operations: $A\oplus B = A+B-J_n(t)$, $A\otimes B = AB-(t-1)J_n(t)$ and $a\times A = aA-(a-1)J_n(t)$, for $A$, $B\in S_n(t)$, $a\in F$, where $A+B$, $AB$ and $aA$ are the usual matrix operations.

**Proof** We note that $AJ_n(t) = AJ_n(1)t = J_n(t)t$ and $J_n(t)B = J_n(t)t$. The rest are straightforward.

From Theorem 4 we can prove the following proposition. E(S) denotes the set of idempotents in a semigroup S.

**Proposition** The number of idempotents in $S_n(t)$ is equal to $\pi = |E(S_n(t))|$:

$$\pi = \sum_{r=1}^{n-1} \frac{[p^{n-1}]}{[p^r][p^{n-1-r}]} + 1, \text{ where } p = |F| \text{ and } [p^m] = (p^m-1)(p^m-p)\cdots(p^m-p^{m-1}).$$

**Proof** We show that $|E(S_n(t))| = |E(S_n(0))|$ for $t\neq 0$. Let $A$ be a member of $S_n(t)$. Then there exists $B$ in $S_n(0)$ such that $A = B+J_n(t)$. If $B$ is idempotent, then $A\otimes A = BB+BJ_n(t)+J_n(t)B+J_n(t)J_n(t)-(t-1)J_n(t) = B+J_n(t) = A$. A similar argument shows that if $A$ is idempotent then $B$ is idempotent. We shall have in Lemma 2 that $S_n(0)$ and $M_{n-1}(F)$ are algebra-isomorphic. Now applying [6, Lamma 5] we obtain that

$$|E(S_n(0))| = |E(M_{n-1}(F))| = \sum_{r=1}^{n-1} \frac{[p^{n-1}]}{[p^r][p^{n-1-r}]} + 1,$$

(the last term 1 refers the zero matrix). This proves the proposition.

**Lemma 2** (1) $S_n(F)$ and $E\oplus M_{n-1}(F)$ are algebra-isomorphic. (2) $S_n(0)$ and $M_{n-1}(F)$ are algebra isomorphic. $F\oplus M_{n-1}(F)$ denotes the direct sum of two algebras F ad $M_{n-1}(F)$.

**Proof** This essentially follows from Theorem 1 and Corollary. (We proved it in [7] by the similar method in the proof of Theorem 1).

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