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On the Relation between Globality and Linearity of Solutions of Analytic O. D. E.'s*

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The purpose of this note is to point out a property of scalar analytic differential equations of the form $w' = f(z, w)$. The property relates the "global existence" of their solutions and the linearity of the mapping $f(z, w)$. Roughly speaking our contribution states that "the global existence of solutions implies the linearity of the mapping $f(z, w)$". It turns out that this property does not hold for vectorial analytic differential systems.

We are ready to formulate

Theorem 1. Let $f(z, w)$ be an analytic function of z and w in the domain $D \times \mathbf{C}$. D is a simply connected domain and $\mathbf{C}$ is the complex plane. The initial value problem.

(1) $w' = f(z, w)$, $w(z_0) = w_0$, $z_0 \in D$

has a ("global") solution $w(z)$ which is analytic for all $z \in D$, for each fixed $z_0 \in D$ and for each fixed $w_0 \in \mathbf{C}$ if and only if

(2) $f(z, w) \equiv A(z)w + B(z)$.

$A(z)$ and $B(z)$ are certain analytic functions of z for $z \in D$.

Proof. First we tackle the trivial part of our claim. Let $f(z, w)$ be of the form given in (2). Then it is well known that $w(z)$ given below

$$w(z) = [\exp A(z)]w_0 + \int_{z_0}^{z} B(s)[\exp \int_s^z A(v)\,dv]\,ds$$

is the unique solution of the initial value problem (1). Therefore for each fixed $z_0 \in D$ and each fixed $w_0 \in \mathbf{C}$, $w(z)$ is an analytic function of z for all $z \in D$. (The integrations above are to be taken along a Jordan curve which is embedded in D and which connects the points z and z_0.)

Denote also by $w(z, z_0, w_0)$ the unique solution $w(z)$ to the initial value prolbem (1) such that $w(z_0, z_0, w_0) = w_0$. This second notation brings out the dependence of a solution on the initial point z_0 and on the initial value w_0. We may consider $w(z, z_0, w_0)$ as a function of the three variables z, z_0, and w_0. By our assumption this function is defined in the set $D \times D \times \mathbf{C}$. A well known theorem, (see e.g. Hille [2] sec. 2.8) guarantees that $w(z, z_0, w_0)$ is

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an analytic function in the set $D \times D \times \mathbf{C}$.

We claim that for each fixed pair $\langle z, z_0 \rangle \in D \times D$ the mapping $w(z, z_0, w_0)$ must be an entire function which is univalent for all $w_0 \in \mathbf{C}$. Assume to the contrary that for some fixed complex numbers $z_1 \in D$ and $z_0 \in D$ we have

(3) $\quad w_1 = wc(z_1, z_0, w_0) = w(z_1, z_0, \hat{w}_0)$

where $w_0 \neq \hat{w}_0$. Then, the initial value problem

(4) $\quad w' = f(z, w), \quad w(z_1) = w_1$

would have two distinct solutions of the form $w_j(z, z_1, w_1)$, $j = 1, 2$. By virtue of (3) they satisfy

(5) $\quad w_1(z_0, z_1, w_1) = w_0 \neq \hat{w}_0 = w_0 w_2(z_0, z_1, w_1)$.

This contradicts the basic existence and uniqueness theorem to initial value problems. Compare e.g. with Hille [3]ch. 3.

Let $g(w)$ be an entire function which is univalent for all $w \in \mathbf{C}$. Then $g(w)$ must be of the form $g(w) \equiv g_1 w + g_0$ where g_1 and g_0 are certain complex numbers. This is so because $w = \infty$ cannot be an essential singularity of $g(w)$. Consequently the Taylor series expansion of $g(w)$ about $w = 0$ must be a polynomial. The degree of this polynomial must then be one in order to guarantee that $g(w)$ is a one to one mapping.

Therefore we must have

(6) $\quad w(z, z_0, w_0) \equiv \alpha(z, z_0) w_0 + \beta(z, z_0)$.

$\alpha(z, z_0)$ and $\beta(z, z_0)$ are certain analytic functions in $D \times D$. From the relation (1) we obtain that

(7) $$w'(z, z_0, w_0) = \frac{\partial w}{\partial z} = \frac{\partial \alpha(z, z_0)}{\partial z} w_0 + \frac{\partial \beta(z, z_0)}{\partial z} \equiv f(z, w(z, z_0, w_0))$$

holds for all $\langle z, z_0, w_0 \rangle$ in $D \times D \times \mathbf{C}$. Let $z = z_0$ in (7). Then for all $\langle z_0, w_0 \rangle \in D \times \mathbf{C}$ we have the identity

(8) $\quad \hat{\alpha}(z_0) w_0 + \hat{\beta}(z_0) \equiv f(z_0, w(z_0, z_0, w_0)) \equiv f(z_0, w_0)$.

The mappings $\hat{\alpha}(z_0)$ and $\hat{\beta}(z_0)$ are defined by

$$\hat{\alpha}(z_0) := \left. \frac{\partial \alpha(z, z_0)}{\partial z} \right|_{z = z_0}, \quad \hat{\beta}(z_0) := \left. \frac{\partial \beta(z, z_0)}{\partial z} \right|_{z = z_0}$$

This concludes the proof of our theorem.

Theorem 1 does not hold for ordinary differential systems in the complex domain. Consider the nonlinear initial value problem

(9) $$\begin{aligned} &w_1' = w_1, \quad w_1(z_0) = v_1, \quad z_0 \in \mathbf{C}, \quad v_1 \in \mathbf{C}, \\ &w_2' = h(w_1) w_2, \; w_2(z_0) = v_2, \; v_2 \in \mathbf{C}. \end{aligned}$$

The mapping $h(w_1)$ is assumed to be an entire function of w_1. Then, for every triplet $\langle z_0, v_1, v_2 \rangle \in \mathbf{C}^3$ the initial value problem (9) possesses a unique solution $\langle w_1(z),\ w_2(z)\rangle$. The mappings $w_j(z)$, $j=1,2$, are entire functions of z. This remark can be generalized to make the following proposition.

Proposition 2. Given the ininitial value problem

$$(10)\qquad \begin{aligned} & w_1' = h_1(z) w_1 B_1(z), \quad w_1(z_0) = v_1, \quad v_1 \in \mathbf{C}, \\ & w_j'(z) = h_j(z,\ w_1 \cdots w_{j-1}) w_j + B_j(z), \ w_j(z_0) = v_j, v_j \in \mathbf{C}, \quad j = 2 \cdots n. \end{aligned}$$

The mappings h_j, $j=2\cdots n$, are analytic functions in $D\times \mathbf{C}^{j-1}$. The mappings $h_1(z)$ and $B_j(z)$, $j=1\cdots n$, are analytic in the simply connected domain D. Then, for each fixed $z_0\in D$ and for each fixed $\langle v_1\cdots v_n\rangle \in \mathbf{C}^n$ the initial value problem (10) possess an unique solution $\langle w_1(z), \cdots, w_n(z)\rangle$. Each mapping $w_j(z)$, $j=1\cdots n$, is an analytic function in D.

Proof. We omit the trivial details.

Let us name a differential system of the form (10) as being "triangular". The following may be an interesting question to answer. What are the vectorial ordinary differential systems

$$(11)\qquad x_j' = q_j(z,\ x_1\cdots x_n), \quad j=1\cdots n$$

whose investigation "can be reduced" via "certain transformation" to the investigation of the "triangular" systems (10)?. An answer to this question will identify n dimensional ordinary differential systems which have solutions which exist "globally".

We remark that our analysis underscores an additional difference between "analytic" differential equations in the complex plane and differential equations whose solutions are real valued mappings which depend on a real variable. Let $f(t,\ y)$ be a continuous vector function which maps $[a,\ b]\times \mathbf{R}^n$ into $\mathbf{R}^n$. Then $f(t,\ y)$ need not be a linear mapping of y in order to guarantee that the initial value problem

$$(12)\qquad y' = f(t,\ y), \quad y(t_0) = y_0, \quad t_0 \in [a,\ b], \quad y_0 \in \mathbf{R}^n$$

has a continuously differentiable soution on the entire interval $[a,\ b]$. It is well known that if $f(t,\ y)$ is nonlinear and also slublinear (namely, $\|f(t,\ y_1)-f(t,\ y_2)\| \geq k\,\|y_1-y_2\|$, k is a fixed number) then each initial value problem (12) has a continuously differentiable solution on the entire interval $[a,\ b]$.

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