关于重节点多元B样条的构造*

翁祖萌

（浙江工学院）

1. 引　言

设x^0, x^1, …, $x^n\in \mathrm{R}^s$是互异的点，$n\geq s$，$\mathrm{vol}_s[x^0, \cdots, x^n]>0$，这里$[x^0, \cdots, x^n]=\{x=\sum_{j=0}^{n}v_jx^j \mid (v_0, \cdots, v_n)\in \mathrm{S}^n\}$，$\mathrm{S}^n=\{(v_0, \cdots, v_n) \mid \sum_{j=0}^{n}v_j=1, v_j\geq 0, j=0, \cdots, n\}$.

以x^i为m_i+1重节点，$m_i\geq 0$，$i=0, \cdots, n$，的多元B样条$M(x \mid (x^0)^{m_0+1}, \cdots, (x^n)^{m_n+1})$由下式定义（见C. A. Micchelli〔1〕）：

$$\int_{\mathrm{R}^s} f(x)M(x \mid (x^0)^{m_0+1}, \cdots (x^n)^{m_n+1})dx=N!\int_{[(x^0)^{m_0+1}, \cdots, (x^n)^{m_n+1}]} f \qquad (1)$$

其中$N=\left(\sum_{i=0}^{n}(m_i+1)\right)-1$，

$$\int_{[(x^0)^{m_0+1}, \quad (x^n)^{m_n+1}]} f:=\int_{\mathrm{S}^N} f\Big(\Big(\sum_{j=0}^{m_0}v_j\Big)x^0+\Big(\sum_{j=m_0+1}^{m_0+m_1+1}v_j\Big)x^1+\Big(\sum_{j=m_0+m_1+2}^{(\sum_{i=0}^{2}(m_i+1))-1}v_j\Big)x^2+$$
$$+\cdots+\Big(\sum_{j=N-m_n}^{N}v_j\Big)x^n\Big)\mathrm{d}v_1\cdots dv_N,$$

$f(x)\in \mathrm{C}_0^0(\mathrm{R}^s)$是任意的（$\mathrm{C}_0^0(\mathrm{R}^s)$是$\mathrm{R}^s$上支集为有界的连续函数集）.

记$\mathrm{S}:=\{\overbrace{x^0, \cdots, x^0}^{m_0+1}, \cdots, \overbrace{x^n\cdots, x^n}^{m_n+1}\}=\{y^0, y^1, \cdots, y^N\}$，中$y^i$允许按重复度重复出现，记$M(x \mid y^0, \cdots, y^N):=M(x \mid (x^0)^{m_0+1}, \cdots, (x^n)^{m_n+1})$. 由定义式（1）知多元B样条与其节点$x^0$, …, x^n（或y^0, …, y^N）的顺序无关.

由于$\mathrm{vol}_s[y^0, \cdots, y^N]=\mathrm{vol}_s[x^0, \cdots, x^n]>0$，故有$\mathrm{R}^N$中的单纯形$\sigma=[z^0, \cdots, z^N]$，其顶点$z^i=(y^i, \cdots)$（即$z^i$的前$s$个坐标与$y^i$的一致），$i=0, \cdots, N$，且$\mathrm{vol}_N\sigma>0$. 由(1)式作变量代换知

$$M(x \mid y^0, \cdots, y^N)=\frac{\mathrm{vol}_{N-s}\{z\in\sigma \mid z_j=x_j, \ j=1, \cdots, s\}}{\mathrm{vol}_N\sigma} \qquad (2)$$

这里$x=(x_1, \cdots, x_s)\in \mathrm{R}^s$, $z=(z_1, \cdots, z_N)\in \mathrm{R}^N$.（2）式右端的值与所选取的$\sigma$无关.

* 1984年6月4日收到.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

由H. Hakopian〔2〕知$M(x\,|\,(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})$是以$[x^0,\cdots,x^n]$为支集的分段多项式函数，我们注意到〔2〕的结果对于重节点B样条都成立.

本文的目的一方面在于给出较〔1〕更一般的积分递推式（见定理1），并由此拓广〔2〕的一个结果（见定理2），另一方面多元B样条关于节点的连续性以及关于节点的方向导数的表达式是至今尚未探讨过的问题. 我们利用定理1得到的表示式（见定理3）解决了这个问题；在$s=1$的情形又得到了L. Schumaker〔3〕的表示式.

2. 递推式

定理1　设$x^0,\cdots,x^n\in R^s$，$\mathrm{vol}_s[x^0,\cdots,x^n]>0$，则

$$M(x\,|\,(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})=\frac{N!}{m_0!(N-m_0-1)!}\int_1^{\infty}t^{-N+S-1}(t-1)^{m_0}M((1-t)x^0+tx\,|\,(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\,dt, \tag{3}$$

其中$N=\left(\sum_{i=0}^{n}(m_i+1)\right)-1$.

证　由定义，

$$\int_{R^s}f(x)M(x\,|\,(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})\,dx=N!\int_{[(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1}]}f$$

根据〔1〕中（15）式的证明，上式右端为

$$\frac{N!}{m_0!}\int_{S^{N-m_0}}v_0^{m_0}f\Big(v_0x^0+\Big(\sum_{j=1}^{m_1+1}v_j\Big)x^1+\Big(\sum_{j=m_1+2}^{\sum_{i=1}^{2}(m_i+1)}v_j\Big)x^2+\cdots+\Big(\sum_{j=N-m_0-m_n}^{N-m_0}v_j\Big)x^n\Big)dv_1\cdots dv_{N-m_0}$$

$$=\frac{N!}{m_0!}\int_0^1\Big[\int_{v_1+\cdots+v_{N-m_0}=h}(1-h)^{m_0}f\Big(x^0+\Big(\sum_{j=1}^{m_1+1}v_j\Big)(x^1-x^0)+\cdots+\Big(\sum_{j=N-m_0-m_n}^{N-m_0}v_j\Big)(x^n-x^0)\Big)dv_2$$

$$\cdots dv_{N-m_0}\Big]dh=\frac{N!}{m_0!}\int_0^1(1-h)^{m_0}h^{N-m_0-1}\Big[\int_{v_1+\cdots+v_{N-m_0}=1}f\Big(x^0+h\Big(\sum_{j=1}^{m_1+1}v_j\Big)(x^1-x^0)+\cdots+$$

$$h\Big(\sum_{j=N-m_0-m_n}^{N-m_0}v_j\Big)(x^n-x^0)\Big)dv_2\cdots dv_{N-m_0}\Big]dh$$

$$=\frac{N!}{m_0!}\int_0^1h^{N-m_0-1}(1-h)^{m_0}\Big[\frac{1}{(N-m_0-1)!}\int_{R^s}f(x^0+hx)M(x\,|\,(x^1-x^0)^{m_1+1},\cdots,(x^n-$$

$$x^0)^{m_n+1})\,dx\Big]dh=\frac{N!}{m_0!(N-m_0-1)!}\int_0^1h^{N-m_0-1}(1-h)^{m_0}h^{-s}\int_{R^s}f(x)M\Big(\frac{x-x^0}{h}\,\Big|\,(x^1-$$

$$x^0)^{m_1+1},\cdots,(x^n-x^0)^{m_n+1}\Big)\,dx\,dh=\frac{N!}{m_0!(N-m_0-1)!}\int_{R^s}f(x)\,dx\int_0^1h^{+N-m_0-S-1}(1-h)^{m_0}\cdot$$

$$M((1-h^{-1})x^0+h^{-1}x\,|\,(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\,dh$$

$$=\frac{N!}{m_0!(N-m_0-1)!}\int_{R^s}f(x)\,dx\int_1^{\infty}t^{-N+m_0+S-1}\Big(1-\frac{1}{t}\Big)^{m_0}M((1-t)x^0+tx\,|\,(x^1)^{m_1+1},\cdots,$$

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$\cdots(x^n)^{m_n+1})dt$ （4）

这里用到多元B样条的下列性质：

（i）$M(x|(x^0+y)^{m_0+1},\cdots,(x^n+y)^{m_n+1})=M(x-y|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})$

（ii）$M(x|(Ax^0)^{m_0+1},\cdots,(Ax^n)^{m_n+1})=\frac{1}{|\det A|}M(A^{-1}x|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})$

这里 A 是 $s\times s$ 非异方阵。

由（1）式及（4）式知，对任意的$f(x)\in C_0^\infty(R^s)$，

$$\int_{R^s}f(x)M(x|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})dx=\int_{R^s}f(x)\frac{N!}{m_0!(N-m_0-1)!}\int_1^{\infty}t^{-N+s-1}(t-1)^{m_0}M((1-t)x^0+tx|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})dt$$

于是（3）成立．定理1证毕．

注1．从上述证明可知定理1中$x^0,\cdots,x^n$可以有重复的点．此时仍用（1）式定义$M(x|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})$。

2．当$m_0=m_1=\cdots=m_n=0$时定理1系C. A. Micchelli〔1〕所证明．

设$\mathrm{vol}_s[x^0,x^1,\cdots,x^n]>0$，$\mathrm{vol}_s[x^1,\cdots x^n]=0$．于是$\mathrm{vol}_{s-1}[x^1,\cdots,x^n]>0$。设$\{y^0,\cdots,y^N\}=\{\overbrace{x^0,\cdots,x^0}^{m_0+1},\cdots,\overbrace{x^n,\cdots,x^n}^{m_n+1}\}$；$z^i$，$i=0,\cdots,N$，如引言中所述；

$L_{s-1}=\{\sum_{i=m_0+1}^{N}\alpha_iy^i|\sum_{i=m_0+1}^{N}\alpha_i=1\}$．对于任意的$x\in L_{s-1}$，可定义

$M_{s-1}(x|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})$：=

$$\frac{\mathrm{Vol}_{N-m_0-1-(s-1)}\{\sum_{i=m_0+1}^{N}\beta_iz^i|\sum_{i=m_0+1}^{N}\beta_iy^i=x,(\beta_{m_0+1},\cdots\beta_N)\in S^{N-m_0-1}\}}{\mathrm{vol}_{N-m_0-1}[z^{m_0+1},z^{m_0+2},\cdots,z^N]} \quad (5)$$

我们有

定理2　设$x^0,x^1,\cdots,x^n\in R^s$，$\mathrm{vol}_s[x^0,\cdots,x^n]>0$，$\mathrm{vol}_s[x^1,\cdots,x^n]=0$．则

$$M(tx+(1-t)x^0|(x^0)^{m_0+1},(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})=\frac{N!}{(N-m_0)!m_0!}Ct^{N-m_0-s}(1-t)^{m_0}\cdot M_{s-1}(x|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1}), \quad (6)$$

其中$0\le t\le1$，$x\in L_{s-1}$，$c=\frac{N-m_0}{\rho(x^0,L_{s-1})}$，$\rho(x^0,L_{s-1})$是$x^0$到$L_{s-1}$的距离．而当$t<0$或$t>1$时（6）式左端等于零．

注．1．在$m_0=0$的情形定理2系H. Hakopian〔2〕所证明．

2．当$n=s$时，反复利用（6）式即得〔1〕中$M(x|(x^0)^{m_0+1},\cdots,(x^s)^{m_s+1})$的表达式．

证　当$t<0$或$t>1$时$tx+(1-t)x^0\notin[x^0,x^1,\cdots,x^n]$，故显然有$M(tx+(1-t)x^0|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})=0$．当$0\le t\le1$时证明如下．

由上述的注只需就$m_0>0$的情形证明．

由定理1及其注1，

$$M(tx+(1-t)x^0|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})=M(tx+(1-t)x^0|(x^0)^{m_0},x^0,(x^1)^{m_1+1},\cdots(x^n)^{m_n+1})=\frac{N!}{(m_0-1)!(N-m_0)!}\int_1^{\infty}\tau^{-N+s-1}(\tau-1)^{m_0-1}M((1-\tau)x^0+\tau(tx+(1-t)x^0)$$

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$|x^0,\ (x^1)^{m_1+1},\ \cdots,\ (x^n)^{m_n+1})d\tau$ （7）

设$x\in L_{s-1}$．由于（$1-\tau)x^0+\tau(tx+(1-t)x^0)=(1-\tau t)x^0+\tau tx$，故当$\tau>\frac{1}{t}$时（7）式右端的被积式为0．从〔2〕关于$m_0=0$情形的结果知：当$1\le\tau\le\frac{1}{t}$时，

$$M((1-t\tau)x^0+t\tau x\,|x^0,\ (x^1)^{m_1+1},\ \cdots,\ (x^n)^{m_n+1})=\frac{N-m_0}{\rho(x^0,\ L_{s-1})}(t\tau)^{N-m_0-s}\times$$

$$M_{s-1}(x\,|(x^1)^{m_1+1},\ \cdots,\ (x^n)^{m_n+1}).\qquad(8)$$

将（8）式代入（7）式，得

$$M(tx+(1-t)x^0\,|(x^0)^{m_0+1},\ \cdots,\ (x^n)^{m_n+1})=\frac{N!c}{(m_0-1)!(N-m_0)!}\int_1^{\frac{1}{t}}\tau^{-N+s-1}\times$$

$$(\tau-1)^{m_0-1}(t\tau)^{N-m_0-s}M_{s-1}(x\,|(x^1)^{m_1+1},\ \cdots,\ (x^n)^{m_n+1})d\tau$$

$$=\frac{N!c}{(m_0-1)!(N-m_0)!}t^{N-m_0-s}M_{s-1}(x\,|(x^1)^{m_1+1},\ \cdots,\ (x^n)^{m_n+1})\int_1^{\frac{1}{t}}\tau^{-m_0-1}(\tau-1)^{m_0-1}d\tau$$

$$=\frac{N!}{m_0!(N-m_0)!}ct^{N-m_0-s}(1-t)^{m_0}M_{s-1}(x\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})$$定理2证毕．

3．关于节点的方向导数

若集$S\subset R^s$（见引言）中任意$s+d$个元（同一点可重复出现，而视为不同的元）的仿射包(affine hull)是R^s，且存在S中$s+d-1$个元，其仿射包是$s-1$维超平面，则称S是d退化的（参见〔2〕）．

记$M(x\,|(x^0)^{m_0+1},\ \cdots,\ (x^n)^{m_n+1})$关于变元$x^i$的$u$方向导数为$D_u^{x^i}\ M(x\,|\ (x^0)^{m_0+1},\ \cdots,\ (x^i)^{m_i+1},\cdots,\ (x^n)^{m_n+1})$，记$D_uM(\cdot\ |(x^0)^{m_0+1},\ \cdots,\ (x^n)^{m_n+1})(x)$为$D_uM(x\,|(x^0)^{m_0+1},\ \cdots,\ (x^n)^{m_n+1})$．

定理3．设$x^0,\ x^1,\ \cdots,\ x^n\in R^s$，$\{x^1,\ \overset{m_1+1}{\cdots},\ x^1,\ \cdots,\ x^n,\ \overset{m_n+1}{\cdots},\ x^n\}$是$d'$退化的，$\sum_{j=1}^{n}(m_j+1)>s+d'$，则$M(x\,|(x^0)^{m_0+1},\ \cdots,\ (x^n)^{m_n+1})$关于变元$x^0$在$R^s$是连续的．若$\sum_{j=1}^{n}(m_j+1)>s+d'+1$，则对于任意方向$u$，

$$D_u^{x^0}M(x\,|(x^0)^{m_0+1},\ \cdots,\ (x^n)^{m_n+1})=-\frac{m_0+1}{N+1}D_uM(x\,|(x^0)^{m_0+2},\ (x^1)^{m_1+1},\ \cdots,\ (x^n)^{m_n+1})\qquad(9)$$

且（9）式右端关于x^0与x在$R^s\times R^s$上是连续的．我们需要下列的

引理（〔2〕）设$S=\{y^0,\ \cdots,\ y^N\}\subset R^s$是$d$退化的，$vol_s[y^0,\ \cdots,\ y^N]>0$．则$M(\cdot\ |y^0,\ \cdots,\ y^N)\in C^{N-s-d}(R^s)\setminus C^{N-s-d+1}(R^s)$．

定理3证明．显然当$\sum_{j=1}^{n}(m_j+1)>s+d'$时，$vol_s[x^1,\ \cdots,\ x^n]>0$．故（3）式成立．

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

由引理知：当$\sum_{j=1}^{n}(m_j+1)>s+d'$时，（3）式右端被积函数关于变元$x^0$在$R^s$是连续的．

由于$\mathrm{supp}M(\cdot\,|(x^1)^{m+1},\cdots,(x^n)^{m_n+1})=[x^1,\cdots,x^n]$，所以对任意$x\in R^s$．

$$\lim_{u\to 0}M(x\,|(x^0+u)^{m_0+1},(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})=\lim_{u\to 0}\frac{N!}{m_0!(N-m_0-1)!}\int_1^{\infty}t^{-N+s-1}(t-1)^{m_0}\cdot$$

$$M((1-t)(x^0+u)+tx\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\mathrm{d}t=\frac{N!}{m_0!(N-m_0-1)!}\int_1^{\infty}t^{-N+s-1}(t-1)^{m_0}M$$

$$M((1-t)x^0+tx\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\mathrm{d}t=M(x\,|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1}).\tag{10}$$

而当$\sum_{j=1}^{n}(m_j+1)>s+d'+1$时，由引理及（3）式知

$$D_u^{x^0}M(x\,|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})=\frac{N!}{m_0!(N-m_0-1)!}\int_1^{\infty}t^{-N+s-1}(t-1)^{m_0}(1-t)\times$$

$$D_uM((1-t)x^0+tx\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\mathrm{d}t=\frac{-N!}{m_0!(N-m_0-1)!}\int_1^{\infty}t^{-N+s-1}(t-1)^{m_0+1}\times$$

$$D_uM((1-t)x^0+tx\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\mathrm{d}t\tag{11}$$

又$$D_uM(x\,|(x^0)^{m_0+2},(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})=D_u\Big[\frac{(N+1)!}{(m_0+1)!(N+1-m_0-2)!}\int_1^{\infty}t^{-N+s-2}\times$$

$$(t-1)^{m_0+1}M((1-t)x^0+tx\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\mathrm{d}t\Big]=\frac{(N+1)!}{(m_0+1)!(N-m_0-1)!}\int_1^{\infty}t^{-N+s-2}\times$$

$$(t-1)^{m_0+1}tD_uM((1-t)x^0+tx\,|(x^1)^{m_1+1},\cdots,(x^n)^{m_n+1})\mathrm{d}t\tag{12}$$

比较（11）、（12）两式知（9）式成立．又由引理及（12）式知（9）式右端关于x^0与x在$R^s\times R$是连续的．定理3证毕．

注1．由于多元B样条是分段多项式，故当$\sum_{j=1}^{n}(m_j+1)=s+d'$（或$\sum_{j=1}^{n}(m_j+1)>s+d'$）时，由上述证明过程知，$M(x|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})$在除出某些$s-1$维面外，关于变元$x^0$是连续的（或可微的）．

2．当$u=x^n-x^0$时，由定理3与〔2〕的定理4知$D_{x^n-x^0}^{x^i}M(x\,|(x^0)^{m_0+1},\cdots,(x^n)^{m_n+1})$

$$=(m_i+1)[M(x\,|(x_0)^{m_0},(x^1)^{m_1+1},\cdots,(x^{i-1})^{m_{i-1}+1},(x^i)^{m_i+2},(x^{i+1})^{m_{i+1}+1},\cdots,$$
$$(x^n)^{m_n+1})-M(x\,|(x^0)^{m_0+1},\cdots,(x^{i-1})^{m_{i-1}+1},(x^i)^{m_i+2},(x^{i+1})^{m_{i+1}+1},\cdots,(x^{n-1})^{m_{n-1}+1},$$
$$(x^n)^{m_n})]\tag{13}$$

在$s=1$的情形，由（13）式即得L.Schumaker〔3〕，定理4.27．

参 考 文 献

〔1〕C. A. Micchelli, On a numerically efficient method for computing multivariate B—splines, in Multivariate Approximation Theory, W. Schempp and K. Zeller, eds, ISNM51, Birkhauser, Basel, 1979·

〔2〕H. Hakopian, Multivariate spline functions, SIAM J. Numer. Anal. 19 (1982), pp. 510–517.

〔3〕L. Schumaker, Spline Functions: Basic Theory, John Wiley& Sons, New York, 1981.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.