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Finite Element Method for a Class of Nonliner Problems I-Abstract Results*

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This paper is devoted to the study of finite dimensional approximation of a class of nonlinear problems. Under some conditions, we show that the approximate solutions are convergent in the cases of branches of nonsingular solutions, limit points and simple bifurcation points. This work establishes the theoretical foundation of conforming element, nonconforming element and quasi-conforming element methods for Navier-Stokes equations and Von Karman's equations.

I. Problems and The Basic Assumptions

Let X be a real Hilbert space with the product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$, $\wedge$ a subset in R^k, $\widetilde{X}$ a subspace of X, and $X_0 \subset \widetilde{X}$ be a closed subspace in $\widetilde{X}$. Assume that $A: X \to X$ is a bounded linear operator and $G: \wedge \times \widetilde{X} \to X$ is a nonlinear operator. Denote T_0 the orthogonal projection operator from X to X_0 and set

$$(\lambda, v) \in \wedge \times \widetilde{X}, \quad F(\lambda, v) = Av + G(\lambda, v), \quad F_0(\lambda, v) = T_0 F(\lambda, v). \tag{1.1}$$

We consider the finite dimensional approximation of the following equation:

$$(\lambda, u) \in \wedge \times X_0, \quad F_0(\lambda, u) = 0. \tag{1.2}$$

For the parameter h in $(0,1)$, we choose a finite dimensional subspace of $\widetilde{X}$, say X_h. Define $T_h: X \to X_h$ the orthogonal projection operator, and for $(\lambda, v) \in \wedge \times \widetilde{X}$, $F_h(\lambda, v) = T_h F(\lambda, v)$. The finite dimensional approximation of (1.2) is the following problems:

$$(\lambda, u_h) \in \wedge \times X_h, \quad F_h(\lambda, u_h) = 0. \tag{1.3}$$

Generally, X_h is not a subspace of X_0, and the problem (1.3) is a "nonconforming" approximation of problem (1.2).

$\{X_h\}$, X_0 is called having the approximability if for every v in X_0, $\lim_{h\to 0} \inf_{v_h \in X_h} \|v - v_h\| = 0$. $\{X_h\}$, X_0 is called to be weakly closed if for every weakly convergent sequence $\{v_m\}$ with v_m in X_{h_m}, $m \in \boldsymbol{N}$ and $h_m \to 0$ as $m \to \infty$, the limit of the sequence is in X_0, where $\boldsymbol{N} = \{1, 2, \cdots\}$.

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Operator A is uniformly X_h elliptic if there exists a constant $\eta > 0$ independent of h, such that,

$$v_h \in X_h, \quad (Av_h, v_h) \geq \eta \|v_h\|^2 . \tag{1.4}$$

This paper is based on the following basic assumptions (H): i) $\{X_h\}$, X_0 has the approximability and is weakly closed; ii) A is uniformly X_h-elliptic; iii) $G: \Lambda \times \widetilde{X} \to X$ is p-times Gateaux differentiable for some $p \geq 2$, and the r'th Gateaux derivative $d^r G$ of G is r-linear for all $r \leq p$; iv) for arbitrary h, h' in $[0, 1)$, $T_h G: \Lambda \times X_{h'} \to X$ is p-times Frechet differentiable, and the set $\{\|T_h d^r G(\lambda, v)\|_{L_r(R^k \times X_h, X)} \mid (\lambda, v) \in B \cap (\Lambda \times X_h),\ h \in [0, 1)\}$ is bounded provided B is a bounded set in $R^k \times X$ and $r \leq p$; and v) if $v_m \in X_{h_m}$ for $m \in N$, v_m weakly converges to 0 and $h_m \to 0$, then $\lim_{m \to \infty} (d_u G(\lambda, u) v_m, v_m) = 0$ for $\forall (\lambda, u)$ in $\Lambda \times X_0$; if $(\lambda_m, v_m) \in \Lambda \times X_{h_m}$ for $m \in N$, $(\lambda_m, v_m) \to (\lambda, v)$ in $\Lambda \times X_0$ as $h_m \to 0$ then $\lim_{m \to \infty} \|T_{h_m}(d^r G(\lambda_m, v_m) - d^r G(\lambda, v))\|_{L_r(R^r \times X_{h_m}, X)} = 0$ for $0 \leq r \leq p - 1$. Here $d^0 G = G$, $d_u G$ is the partial Gateaux derivative of G with respect to u, and $\|\cdot\|_{L_0(Y, Z)} = \|\cdot\|_Z$ and $\|\cdot\|_Z$ is the norm of Z for Banach spaces Y and Z (see [1]).

Remark. Actually, we want to discuss such a kind of nonlinear problems that A is X_0-elliptic and $T_0 G: X_0 \to X_0$ is p-times Frechet differentiable and compact, and $T_0 d^r G: \Lambda \times X_0 \to L_r(R^k \times X_0, X_0)$ $(r \geq 1)$ are bounded operators. In this case (H) holds if $X_h \subset X_0$ and $\{X_h\}$, X_0 has the approximability. When $X_h \not\subset X_0$, X_h must possess some properties similar to those of X_0 if we want to get convergent approximate solutions. We shall see that the assumptions (H) are enough for this purpose.

2. Branches of Nonsingular Solutions

In this section, let $\{(\lambda, u(\lambda)) \mid \lambda \in \Lambda\}$ be a branch of nonsingular solutions of the equation (1.2), i.e., $F_0(\lambda, u(\lambda)) = 0$ and $d_u F_0(\lambda, u(\lambda)): X_0 \to X_0$ is an isomorphism for $\lambda \in \Lambda$.

Theorem 1. Let (H) hold and $\{(\lambda, u(\lambda)) \mid \lambda \in \Lambda\}$ be a branch of nonsingular solutions of equation (1.2). Assume Λ is closed and bounded. Then there exists an unique C^p mapping $u_h: \Lambda \to X_h$, for h sufficiently small, such that,

$$\begin{cases} F_h(\lambda, u_h(\lambda)) = 0, \\ \lim_{h\to 0} \sup_{\lambda\in\wedge} \sum_{r=0}^{p-1} \| d^r(u(\lambda) - u_h(\lambda)) \|_{L_r(R^k, X)} = 0. \end{cases} \tag{2.1}$$

Proof. First, we show that there exists h_0 in $(0,1)$ independent of λ and a constant C independent of h and λ, such that,

$$\inf_{0\neq v\in X_h} \sup_{0\neq w\in X_h} |(d_uF_h(\lambda, T_hu(\lambda))v, w)| / \|v\| \|w\| \geq C > 0, \tag{2.2}$$

is true for $h \leq h_0$ and λ in $\wedge$. Otherwise, for each m in $\boldsymbol{N}$, there exists λ_m in $\wedge$ and v_m in X_{h_m} such that, $h_m \to 0$ as $m \to \infty$ and

$$m \in \boldsymbol{N}, \quad 1 = \|v_m\| > m \sup_{0\neq w\in X_{h_m}} |(d_uF_{h_m}(\lambda_m, T_{h_m}u(\lambda_m))v_m, w)| / \|w\|. \tag{2.3}$$

By the weak closedness of $\{X_h\}$, X_0, we can choose a subsequence $\boldsymbol{N}'$ of $\boldsymbol{N}$ and v_0 in X_0, such that $\{v_m\}_{m\in\boldsymbol{N}'}$ weakly convergent v_0 and $\{\lambda_m\}_{m\in\boldsymbol{N}'}$ converges to λ in $\wedge$. By the approximability of $\{X_h\}$, X_0, $T_{h_m}u(\lambda_m)$ converges to $u(\lambda)$ and $T_{h_m}v$ converges to v for $\forall v$ in X_0. Thus (2.3) leads to $(Av_m, T_{h_m}v) + (d_uG(\lambda_m, T_{h_m}u(\lambda_m))v_m, T_{h_m}v) \to 0$ as $m \to \infty$ and m in $\boldsymbol{N}'$. Noticing v) $(d_uG(\lambda_m, T_{h_m}u(\lambda_m))v_m, T_{h_m}v) \to (d_uG(\lambda, u(\lambda))v_0, v)$ as $m \to \infty$ and $m \in \boldsymbol{N}'$. It follows from $(Av_m, T_{h_m}v) \to (Av_0, v)$ that $(d_uF_0(\lambda, u(\lambda))v_0, v) = 0$. Thus $v_0 = 0$. On the other hand, (2.3) gives that $(Av_m, v_m) + (d_uG(\lambda_m, T_{h_m}u(\lambda_m))v_m, v_m) \to 0$ as $m \to \infty$ and $m \in \boldsymbol{N}'$. Again by v) in (H), we get that $(d_uG(\lambda_m, T_{h_m}u(\lambda_m))v_m, v_m) \to 0$. It follows from ii) in (H) that $\lim_{m\to\infty, m\in\boldsymbol{N}'} \|v_m\| = 0$. This is contradictory that $\|v_m\| = 1$.

Second, we show that

$$\|T_hu(\lambda) - T_hu(\lambda^*)\| \leq C\|\lambda - \lambda^*\|, \quad \forall \lambda, \lambda^* \in \wedge, \tag{2.4}$$

$$\lim_{h\to 0} \sup_{\lambda\in\wedge} \sum_{r=0}^{p-1} \|d^r(u(\lambda) - T_hu(\lambda))\|_{L_r(R^k, X)} = 0. \tag{2.5}$$

where C is a constant independent of h and λ, λ^*.

Inequality (2.4) is obvious from the continuity of $u(\lambda)$ and the boundedness of T_h.

For $\forall \varepsilon > 0$, there exist $\lambda_1 \cdots, \lambda_n$ in $\wedge$ such that,

$\min_{1\le i\le n}\|u(\lambda)-u(\lambda_i)\|\le\varepsilon/3$ for $\lambda\in\wedge$. And by the appro of $\{X_h\}$, X_0, there exists $\tilde{h}$ only dependent on ε and λ, such that $\|u(\lambda_i)-T_hu(\lambda_i)\|<\varepsilon/3$, $h\in(0,\tilde{h})$, $1\le i\le n$. Hence for h in $(0,\tilde{h})$ and $\lambda\in\wedge$, we have $\|u(\lambda)-T_hu(\lambda)\|<\varepsilon$. Thus we get $\lim_{h\to 0}\sup_{\lambda\in\wedge}\|u(\lambda)-T_hu(\lambda)\|=0$.

Noticing $d^rT_h=T_hd^r$, we can prove

$$\lim_{h\to 0}\sum_{r=1}^{p-1}\sup_{\lambda\in\wedge}\|d^r(u(\lambda)-T_hu(\lambda))\|_{L_r(R^k,X)}=0,$$

by the similar way. Hence (2.5) is proved to be true.

Thirdly, we show that

$$\lim_{h\to 0}\sup_{\lambda\in\wedge}\|F_h(\lambda,T_hu(\lambda))\|=0. \tag{2.6}$$

For $\forall\varepsilon>0$, let $\lambda_1,\cdots,\lambda_n$ in $\wedge$ satisfy

$$\sup_{\lambda\in\wedge}\min_{1\le i\le n}(\|u(\lambda)-u(\lambda_i)\|+\|\lambda-\lambda_i\|)<\varepsilon.$$

Then for $\forall\lambda\in\wedge$ we have

$$\|F_h(\lambda,T_hu(\lambda))\|\le\min_{1\le i\le n}\{\|F_h(\lambda,T_hu(\lambda))-F_h(\lambda_i,T_hu(\lambda_i))\| \\ +\|F_h(\lambda_i,T_hu(\lambda_i))-F_h(\lambda_i,u(\lambda_i))\|+\|F_h(\lambda_i,u(\lambda_i))\|\}. \tag{2.7}$$

$\{(\lambda,T_hu(\lambda))|\lambda\in\wedge,h\in(0,1)\}$ is a bounded set in $R^k\times X$ because $\{u(\lambda)|\lambda\in\wedge\}$ is a bounded set in X and T_h is the orthogonal projection operator. It foilows that

$$\min_{1\le i\le n}\|F_h(\lambda,T_hu(\lambda))-F_h(\lambda_i,T_hu(\lambda_i))\|<M\varepsilon, \tag{2.8}$$

where M is a constant independent of h and λ. By v) in (H), there exists h' only dependent on λ_i and ε, such that,

$$\max_{1\le i\le n}\|F_h(\lambda_i,T_hu(\lambda_i))\ F_h(\lambda_i,u(\lambda_i))\|<\varepsilon. \tag{2.9}$$

is true for all h in $(0,h')$.

For the last term of the right hand of inequality (2.7), we have

$$\lim_{h\to 0}\|F_h(\lambda_i,u(\lambda_i))\|=0,\quad 1\le i\le n. \tag{2.10}$$

In fact, let h_m be a sequence such that $h_m\to 0$, and

$\lim_{h\to 0}\sup_{\xi\le h}\|F_\xi(\lambda_i,u(\lambda_i))\|=\lim_{m\to\infty}\|F_{h_m}(\lambda_i,u(\lambda_i))\|$, and choose w_m in X_{h_m} satisfying $\|w_m\|=1$, $m\in N$, and

$$\lim_{m\to\infty}\|F_{h_m}(\lambda_i, u(\lambda_i))\| = \lim_{m\to\infty}(F(\lambda_i, u(\lambda_i)), w_m).$$

By the weak closedness of $\{X_h\}$, X_0, there exists subsequence N' of N and w_0 in X_0, such that $\{w_m\}_{m\in N'}$ weakly converges to w_0. It follows that

$\lim_{m\to\infty,\ m\in N'}(F(\lambda_i, u(\lambda_i)), w_m) = (F(\lambda_i, u(\lambda_i)), w_0) = 0$. Hence (2.10) is true

Thus we can choose h'', such that, $\max_{1\le i\le n}\|F_h(\lambda_i, u(\lambda_i))\| < \varepsilon$ for $h \le h''$. Set $\tilde{h} = \min\{h', h''\}$, we have $\|F_h(\lambda, T_h u(\lambda))\| < (M+2)\varepsilon$ for $h \le \tilde{h}$. The equality (2.6) is proved.

Finally, we show the conclusions of theorem 1. By (2.2), (2.4), (2.5) and (2.6) and (iii) in (H), we can apply the theorem 1 in [2] and get that there exists an unique C^p mapping $\lambda \to u_h(\lambda) \in X_h$ for h sufficiently small, such that, $F_h(\lambda, u_h(\lambda)) = 0$. and $\lim_{h\to 0}\sup_{\lambda\in\Lambda}\|u(\lambda) - u_h(\lambda)\| = 0$. By theorem 2 in paper [2], we can conclude that for $1 \le l \le p-1$.

$$\|d^l(u(\lambda) - u_h(\lambda))\|_{L_l(R^k, X)} \le \|d^l(u(\lambda) - T_h u(\lambda))\|_{L_l(R^k, X)} + C\sum_{r=0}^{l}\|d^r F_h(\lambda, T_h u(\lambda))\|_{L_r(R^k, X)}, \tag{2.11}$$

where C is a constant independent of λ and h. By the above method, we can prove $\lim_{h\to 0}\sup_{\lambda\in\Lambda}\sum_{r=0}^{p-1}\|d^r F_h(\lambda, T_h u(\lambda))\|_{L_r(R^k, X)} = 0$.

Theorem 1 is proved.

3. Singular Solutions

In this section, assume that $T_0 G: X_0 \to X_0$ is a compact operator, and let $(\lambda_0, u_0) \in \Lambda \times X_0$ is a singular point of F_0, i.e., i) $F_0(\lambda_0, u_0) = 0$. and ii) $d_u F_0^0 \equiv d_u F_0(\lambda_0, u_0) \in L_1(X_0, X_0)$ is not an isomorphism from X_0 to X_0. We want to solve equation (1.2) in a neighborhood of (λ_0, u_0). Let δ_{ij} be the Kronecker's delta, sgn (y) be the sign function, i.e., sgn $(y) = y/|y|$ for $y \ne 0$ and sgn$(0) = 0$.

Lemma 1. There exists an integer r in N and $\varphi_{i,0}, \varphi^*_{i,0}$ in X_0, $1 \le i \le r$, such that, I) $d_u F_0^0 \varphi_{i,0} = 0$, $(d_u F_0^0)^* \varphi^*_{i,0} = 0$, $(\varphi_{i,0}, \varphi^*_{j,0}) = \delta_{ij}$, $1 \le i, j \le r$; II) let X_0^1 Be the Kernal of $d_u F_0^0: X_0 \to X_0$ and $X_0^2 = d_u F_0^0 X_0$, then $X_0 = X_0^1 + X_0^2$ and X_0^1 is the space spanned by $\{\varphi_{1,0}, \cdots, \varphi_{r,0}\}$ and $X_0^2 = \{v \mid v \in X_0, (v, \varphi^*_{i,0}) = 0, 1 \le i \le r\}$, and III) $d_u F_0^0$ is an isomorphism from X_0^2 to X_0^2.

Proof. Because A is X_0-elliptic and $T_0 G$ is compact, $T_0 A + d_u T_0 G$ is a Fredholm operator of index zero. We can immediately get lemma 1 by the theory of Fredholm operator.

Denote, for $h\in(0,1)$,

$$\begin{cases}\varphi_{1,h}=T_h\varphi_{1,0},\quad \varphi^*_{1,h}=T_h\varphi^*_{1,h},\\ \varphi_{j,h}=T_h\varphi_{j,h}-\sum_{i=1}^{j-1}(T_h\varphi_{j,0},\varphi^*_{i,h})\,\varphi_{i,h}/(\varphi_{i,h},\varphi^*_{i,h}),\quad 2\le j\le r.\\ \varphi^*_{j,h}=T_h\varphi^*_{j,0}-\sum_{i=1}^{j-1}(T_h\varphi^*_{j,0},\varphi_{i,h})\,\varphi^*_{i,h}/(\varphi_{i,h},\varphi^*_{i,h}),\end{cases}\tag{3.1}$$

From the approximability of $\{X_h\}$, X_0, we know that $\varphi_{j,h}$, $\varphi^*_{j,h}$ are well-defined when h is sufficiently small and that

$\lim_{h\to 0}\{\|\varphi_{j,h}-\varphi_{j,0}\|+\|\varphi^*_{j,h}-\varphi^*_{j,0}\|\}=0$ with $1\le j\le r$. Thus we have

Lemma 2. $\lim_{h\to 0}\sum_{i=1}^{r}(\|\varphi_{i,0}-\varphi_{i,h}\|+\|\varphi^*_{i,0}-\varphi^*_{i,h}\|)=0$ and for h sufficiently small, sgn $(\varphi_{i,h},\varphi^*_{j,h})=\delta_{ij}$, $1\le i,j\le r$.

For the sake of convenience, assume that the conclusion of lemma 2 is true for all $h\in(0,1)$. Then for $h\in[0,1)$, denote

$X^1_h=\{v|v=\sum_{i=1}^{r}c_i\varphi_{i,h}, c_i\in R\}$ and $X^2_h=\{v|v\in X_h,\ (v,\varphi^*_{j,h})=0.\ 1\le j\le r\}$.

And for v in X, define

$$Q_hv=v-\sum_{i=1}^{r}(v,\varphi^*_{i,h})\varphi_{i,h}/(\varphi_{i,h},\varphi^*_{i,h}).\tag{3.2}$$

Lemma 3. $X_h=X^1_h+X^2_h$ for $h\in(0,1)$ and $\{X^2_h\}$, X^2_0 has the approximability and the weak closedness.

Proof. Definition (3.2) tells us that every v in X_h can be expressed as a sum of an element in X^1_h and one in X^2_h. Now we show that this expression is unique. Let $v+w=0$ with $u=\sum_{i=1}^{r}c_i\varphi_{i,h}$ and w in X^2_h. It follows from w in X^2_h that $(u,\varphi^*_{j,h})=0$ for $1\le j\le r$. On the other hand, $(u,\varphi^*_{j,h})=c_j$. Thus $u=0$, therefor $w=0$. So $X_h=X^1_h+X^2_h$.

For every φ in X^2_0, $\varphi-T_h\varphi=\varphi-Q_hT_h\varphi+(Q_h-I)T_h\varphi$ with I identify operator. By i) in (H), $\lim\|\varphi-T_h\varphi\|=0$. And $\lim(Q_h-I)T_h\varphi=\lim\sum_{i=1}^{r}(T_h\varphi,\varphi^*_{i,h})\varphi_{i,h}/(\varphi_{i,h},\varphi^*_{i,h})=0$. Thus $\lim(\varphi-Q_hT_h\varphi)=0$. The approximability holds.

If $\varphi_m \in X^2_{h_m}$, $m \in N$, and $h_m \to 0$ and φ_m weakly converges to φ_0 as $m \to \infty$, then $\varphi_0 \in X_0$, And $0 = \lim(\varphi_m, \varphi^*_{j,h_m}) = (\varphi_0, \varphi^*_{j,0})$, $1 \le j \le r$. Hence $\varphi_0 \in X^2_0$. The weak closedness is true.

According to lemmas 1 and 3, the problems (1.2) and (1.3) are equivalent to the following problems respectively: $h \in [0,1)$,

$$Q_h F_h(\lambda, u_h) = 0 \text{ and } (I - Q_h) F_h(\lambda, u_h) = 0 . \tag{3.3}$$

Now we define, for v in X and $(\xi, a) \in R^k \times R^r$, $h \in [0,1)$,

$$\mathscr{F}_h(\xi, a, v) = Q_h F_h(\lambda_0 + \xi, T_h u_0 + a^T \Phi_h + v) , \tag{3.4}$$

where $\Phi_h = (\varphi_{1,h}, \cdots, \varphi_{r,h})^T$, Then the first equation of (3.3) becomes

$$v_h \in X^2_h, \quad \mathscr{F}_h(\xi, a, v_h) = 0 . \tag{3.5}$$

Set $S^\rho_m = \{x | x \in R^m, \ \|x\| \le \rho\}$ for $\rho > 0$ and $m \in N$. By lemma 1 and the implicit function theorem, we get a positive number ρ and an unique C^p mapping v_0: $S^\rho_k \times S^\rho_r \to X^2_0$, such that,

$$\begin{cases} \mathscr{F}_0(\xi, a, v_0(\xi, a)) = 0, \quad v_0(0,0) = 0, \\ d_v \mathscr{F}_0(\xi, a, v_0(\xi, a)) \text{ is an isomorphism from } X^2_0 \text{ to } X^2_0 . \end{cases} \tag{3.6}$$

By the way used in section 2, we have the following results.

Theorem 2. Assume that (H) holds and $T_0 G$: $X_0 \to X_0$ is compact and that (λ_0, u_0) is a singular point of F_0, then there is, for h sufficiently small, an unique C^p mapping v_h: $S^\rho_k \times S^\rho_r \to X^2_h$ satisfying

$$\begin{cases} \mathscr{F}_h(\xi, a, v_h(\xi, a)) = 0, \\ \lim\limits_{h \to 0} \sup\limits_{(\xi, a) \in S^\rho_k \times S^\rho_r} \sum\limits_{i=0}^{p-1} \| d^i_{(\xi,a)} (v_0(\xi, a) - v_h(\xi, a)) \|_{L_i(R^{k+r}, X)} = 0 . \end{cases} \tag{3.7}$$

For the convenience' sake, assume the conclusion of theorem 2 is true for all $h \in (0,1)$. Then solving equation (3.3) in a neighborhood of (λ_0, u_0) amounts to solve the following bifurcation equations: $h \in [0,1)$,

$$f_h(\xi, a) \equiv \begin{bmatrix} (F(\lambda_0 + \xi, T_h u_0 + a^T \Phi_h + v_h(\xi, a)), \varphi^*_{1,h}) \\ \vdots \\ (F(\lambda_0 + \xi, T_h u_0 + a^T \Phi_h + v_h(\xi, a)), \varphi^*_{r,h}) \end{bmatrix} = 0 . \tag{3.8}$$

It is easy to verify

$$f_0(0,0) = 0 \text{ and } d_a f_0(0,0) = 0 .$$

Now the approximation of the solutions of equation (1.3) amounts to the approximation problem of the solutions of equation (3.8). In this paper, we will discuss the cases of limit points and simple bifurcation points. Other cases require further work. By the way in section 2, we can show the following results.

Lemma 4. Let the assumptions in theorem 2 hold. Then for $i=0,\cdots,p-1$,

$$\lim_{h\to 0}\sup_{(\xi,a)\in S_k^\rho\times S_r^\rho}\|d^i_{(\xi,a)}(f_0(\xi,a)-f_h(\xi,a))\|_{L_i(R^{k+r},R^r)}=0. \qquad (3.9)$$

4. Limit points and Simple Bifurcation points

First let $(\lambda_0,u_0)\in\Lambda\times X_0$ is a limit point of F_0, i.e., (λ_0,u_0) is a singular point of F_0 and

$$\operatorname{rank}(d_\xi f_0(0,0))=r. \qquad (4.1)$$

For convenience' sake, set $\xi=(\theta,\zeta)^T$, $\theta=(\theta_1,\cdots,\theta_r)$ and $\operatorname{rank}(d_\theta f_0(0,0))=r$. Denote, for $h\in[0,1)$, $(\theta,\zeta)^T\in S_k^\rho$, $a\in S_r^\rho$, $f_h(\theta,\zeta,a)=f_h((\theta,\zeta)^T,a)$. Again by implicit function theorem, we get $\rho'>0$ and a unique C^p mapping $\theta_0: S_k^{\rho'}\to R^r$ satisfying

$$\begin{cases} f_0(\theta_0(\zeta,a),\zeta,a)=0,\ \theta_0(0,0)=0,\\ d_\theta f_0(\theta_0(\zeta,a),\zeta,a) \text{ is an isomorphism from } R^r \text{ to } R^r. \end{cases} \qquad (4.2)$$

By lemma 4 and theorem 1 in [3], we can conclude

Lemma 5. Let the assumptions in theorem 2 hold and (4.1) true. Then, for h sufficiently small, there exists an unique C^p mapping $\theta_h: S_k^{\rho'}\to R^r$ satisfying, $f_h(\theta_h(\zeta,a),\zeta,a)=0$ and

$$\lim_{h\to 0}\sum_{i=0}^{p-1}\sup_{(\zeta,a)\in S_k^{\rho'}}\|d^i_{(\zeta,a)}(\theta_0(\zeta,a)-\theta_h(\zeta,a))\|_{L_i(R^k,R^r)}=0. \qquad (4.3)$$

Theorem 2 and lemma 5 lead to the following results.

Theorem 3. Suppose that (H) holds and $T_0G: X_0\to X_0$ is compact, and that (λ_0,u_0) is a limit point of F_0. Then the following statements are true: 1) in a neighborhood of (λ_0,u_0) there is a branch of solutions of equation (1.2), say $\{(\lambda(t),u(t))\,|\,t\in S_1^{\rho'}\}$, satisfying $F_0(\lambda(t),u(t))=0$ and $(\lambda(0),u(0))=(\lambda_0,u_0)$, and 2) for h sufficiently small, there is an unique branch of solutions of equation (1.3), say $\{(\lambda_h(t),u_h(t))\,|\,t\in S_k^{\rho'}\}$, satisfying

$$F_h(\lambda_h(t),u_h(t))=0,\quad t\in S_k^{\rho'}.$$

$$\lim_{h\to 0}\sum_{i=0}^{p-1}\sup_{t\in S_k^{\rho}}\{\|d_t^i(\lambda(t)-\lambda_h(t))\|_{L_i(R^k,R^r)}+\|d_t^i(u(t)-u_h(t))\|_{L_i(R^k,X)}\}=0. \tag{4.4}$$

Remark. If $k=r=1$ and (λ_0,u_0) is nondegenerate turning point, i.e., (λ_0,u_0) is a limit point of F_0 and

$$(d^2_{uu}F_0(0,0)(\varphi_{1,0},\varphi_{1,0}),\varphi^*_{1,0})\neq 0, \tag{4.5}$$

then equation (1.3) has a nondegenerate turning point (λ_h^0,u_h^0), provided (H) holds for $p\geq 3$ and h is sufficiently small. And $\lim\{|\lambda_0-\lambda_h^0|+\|u_0-u_h^0\|\}=0$. This result can be proved by the way used in [3].

Finally, we discuss the case of simple bifurcation points. Let (λ_0,u_0) is a singular point of F_0 and

$$k=r=1,\ d_\lambda F_0(\lambda_0,u_0)\in d_u F_0^0 X_0. \tag{4.6}$$

Under these condintions, f_0 satisfies, $f_0(0,0)=d_a f_0(0,0)=d_\xi f_0(0,0)=0$, that is, $(0,0)$ is a critical point of f_0. Set $C_0=d^2_{\xi\xi}f_0(0,0)$, $B_0=d^2_{\xi a}f_0(0,0)$, $A_0=d^2_{aa}f_0(0,0)$. Call (λ_0,u_0) a simple bifurcation point of F_0, if $B_0^2-A_0C_0>0$

From [4], we know that in a neighborhood of (λ_0,u_0), the solutions of equation (1.2) consist of two C^{p-2} branches which intersect transversally at the point (λ_0,u_0), and they can be parametrized in the following way, $i=1,2$.

$$\lambda_i(t)=\lambda_0+\xi_i(t),\ u_i(t)=u_0+a_i(t)\varphi_{1,0}+v(\xi_i(t),a_i(t)),\ |t|\leq t_0, \tag{4.7}$$

where $\xi_i(t)=t\sigma_i(t)$, $a_i(t)=t\delta_i(t)$, and $\xi_i(t)$ and $a_i(t)$ are C^{p-2} functions, t_0 is a positive number.

Similar to paper [4], we have

Theorem 4. Assume that (H) holds with $p\geq 4$ and T_0G: $X_0\to X_0$ is compact, and that (λ_0,u_0) is a simple bifurcation point of F_0. Then there is a neighborhood U of (λ_0,u_0) in $R\times X$, such that, for h sufficiently small, the set $\mathscr{S}_h$ of the solutions of (1.3) contained in U consists of two C^{p-2} branches. If these two branches intersect at a point (λ_h^0,u_h^0) in U, they can be parametrized in the form $\{(\lambda_h^i(t),u_h^i(t))\,|\,|t|\leq t_0\}$, $i=1,2$, satisfying

$$\begin{cases}(\lambda_h^i(0),u_h^i(0))=(\lambda_h^0,u_h^0),\quad i=1,2,\\ \lim\limits_{h\to 0}\sum\limits_{j=0}^{p-3}\sup\limits_{|t|\leq t_0}\{|d^j(\lambda_i(t)-\lambda_h^i(t))|+\|d^j(u_i(t)-u_h^i(t))\|\}=0.\quad i=1,2.\end{cases} \tag{4.8}$$

otherwise, the set $\mathscr{S}_h$ is C^{p-2} diffeomorphic to (a part of) a nondegenerate hyperbola and the distance between $\mathscr{S}_h$ and the set $\mathscr{S}$ of the solutions of (1.2) contained in U converges to 0 as $h \to 0$.

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