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On Mean Value Theorems in Guasidifferential Calculus*

Z. Q. Xia
(Department of Applied Mathematics, D.I.T., Dalian, China)

In this note mean value theorems for quasidifferentiable functions initiated by Demyanov and Rubinov are represented, Demyanov et al. (1980, 1981, 1983).

Suppose that a function $f: \mathbf{R} \to \mathbf{R}$ is quasidifferentiable. Given x and $y \in \mathbf{R}$. Let $x < y$ and define an auxiliary function

$$q(t) \triangleq f(t) - (f(y) - f(x))\ (t-x)/\ (y-x),$$

where $t \in \mathrm{cl}\ (x, y) \subset \mathbf{R}$. Obviously, $q(t)$ is continuous and $q(x) = q(y) = f(x)$. So the function $q(t)$ attains its extremum at an interior point $\xi \in (x, y)$. Without loss of generality assume that $q(\xi)$ is the minimum of q on $\mathrm{cl}\,(x, y)$. From a necessary condition for a minimum

$$-\overline{\partial} q(\xi) \subset \underline{\partial} q(\xi),$$

due to Polyakova (1981) and the formula

$$\begin{aligned} Dq(\xi) &= Df(\xi) - (f(y) - f(x))\ [D(t-x)]_\xi/(y-x) \\ &= [\underline{\partial} f(\xi), \overline{\partial} f(\xi)] - ((f(y) - f(x))/\ (y-x))\ [1,\ 0] \\ &= \begin{cases} [\underline{\partial} f(\xi) - \zeta,\ \overline{\partial} f(\xi)], & \text{if } \zeta < 0, \\ [\underline{\partial} f(\xi),\ -\zeta + \overline{\partial} f(\xi)], & \text{if } \zeta \geq 0, \end{cases} \end{aligned}$$

where $\zeta = (f(y) - f(x))/\ (y-x)$, it follows that

$$(f(y) - f(x))/\ (y-x) - \overline{\partial} f(\xi) \subset \underline{\partial} f(\xi).$$

Thus there exist $u \in \underline{\partial} f(\xi)$ and $w \in \partial f(\zeta)$ such that

$$f(y) - f(x) = (u+w)(y-x), \tag{1}$$

where $\xi \in (x, y)$. Of course, the following inclusion relation holds

$$f(y) - f(x) \in (\underline{\partial} f(\xi) + \overline{\partial} f(\xi))\ (y-x). \tag{2}$$

From (2) one has

$$f(y) - f(x) \in \mathrm{co} \bigcup_{\xi \in (x,y)} (\underline{\partial} f(\xi) + \overline{\partial} f(\xi))\ (y-x). \tag{3}$$

We have obtained the following result similar to that by Hiriart-Urruty (1983).

Theorem 1 Suppose $f: \mathbf{R} \to \mathbf{R}$ is quasidifferentiable. Then for any $x, y \in \mathbf{R}$ there exist $\xi \in (x, y)$, $u \in \underline{\partial} f(\xi)$ and $w \in \overline{\partial} f(\xi)$ such that formula (1) holds. In addition, formulae (2) and (3) hold. ∎

Now let us consider the case where the function f is quasidifferentiable on $\mathbf{R}^n$, i.e., $f: \mathbf{R}^n \to \mathbf{R}$. Like above, an auxiliary function is introduced as follows

$$q(t) = q^0(t) - (f(y) - f(x))t,$$

where x and y are fixed, $t \in \mathrm{cl}(0, 1)$ and $q^{\circ}(t) = f(x+t(y-x))$. Since $q^{\circ}(t): \mathbf{R} \to \mathbf{R}$ is quasidifferentiable on $\mathbf{R}$, one has

$$q^{\circ\prime}(t; k) = \lim_{\lambda \to 0^+} (q^{\circ}(t+\lambda k) - q^{\circ}(t))/\lambda$$

$$= \max_{m \in \underline{\partial} q^{\circ}(t)} mk + \min_{n \in \bar{\partial} q^{\circ}(t)} nk,$$

where $k \in \mathbf{R}$. On the other hand,

$$q^{\circ\prime}(t; k) = \lim_{\lambda \to 0^+} (f(x+t(y-x)+\lambda k(y-x)) - f(x+t(y-x)))/\lambda$$
$$= f'(x+t(y-x); k(y-x))$$
$$= \max_{u \in \underline{\partial} f(x+t(y-x))} (k\langle u, y-x\rangle) + \min_{w \in \bar{\partial} f(x+t(y-x))} (k\langle w, y-x\rangle).$$

Therefore we obtain a quasidifferential $Dq^{\circ}(t)$ as follows

$$\underline{\partial} q^{\circ}(t) = \{m = \langle u, y-x\rangle \mid u \in \underline{\partial} f(x+t(y-x))\}$$

and

$$\bar{\partial} q^{\circ}(t) = \{n = \langle w, y-x\rangle \mid w \in \bar{\partial} f(x+t(y-x))\}.$$

The function $q(t)$ is quasidifferentiable with respect to t; of course, it is continuous on interval $\mathrm{cl}(0, 1)$. Just as discussed in Th.1, there exists $t' \in (0, 1)$ such that

$$-\bar{\partial} q(t') \subset \underline{\partial} q(t')$$

if we assume that $q(t)$ attains its least value at $t' \in (0, 1)$. We now find a quasidifferential $Dq(t)$. As before one has

$$Dq(t) = Dq^{\circ}(t) - (f(y) - f(x))Dt$$
$$= [\underline{\partial} q^{\circ}(t), \bar{\partial} q^{\circ}(t)] - (f(y) - f(x))[1, 0]$$
$$= \begin{cases} [\underline{\partial} q^{\circ}(t) - (f(y) - f(x)), \bar{\partial} q^{\circ}(t)], & \text{if } f(y) - f(x) < 0, \\ [\underline{\partial} q^{\circ}(t), \bar{\partial} q^{\circ}(t) - (f(y) - f(x))], & \text{if } f(y) - f(x) \geq 0. \end{cases}$$

In any case, we have the following inclusion relation

$$f(y) - f(x) - \bar{\partial} q^{\circ}(t') \subset \underline{\partial} q^{\circ}(t').$$

So there exist $m \in \underline{\partial} q^{\circ}(t')$ and $n \in \bar{\partial} q^{\circ}(t')$ such that

$$f(y) - f(x) = m + n.$$

It follows from this that there exist $u \in \underline{\partial} f(x+t'(y-x))$ and $w \in \bar{\partial} f(x+t'(y-x))$ such that

$$f(y) - f(x) = \langle u, y-x\rangle + \langle w, y-x\rangle = \langle u+w, y-x\rangle.$$

Let $\varsigma = x + t'(y-x)$, $t' \in (0, 1)$, then the following mean value theorem can be represented, similar to that by Lebourg (1975).

Theorem 2 Suppose $f: \mathbf{R}^n \to \mathbf{R}$ is quasidifferentiable. Then for any $x, y \in \mathbf{R}^n$ there exist $\varsigma \in (x, y)$, $u \in \underline{\partial} f(\varsigma)$ and $w \in \bar{\partial} f(\varsigma)$ such that

$$f(y) - f(x) = \langle u+w, y-x\rangle. \blacksquare$$

Likewise, we have the forms similar to (2) and (3), i.e., $f(y) - f(x) \in \langle \underline{\partial} f(\varsigma) + \bar{\partial} f(\varsigma), y-x\rangle$, where $\varsigma \in (x, y)$ and $f(y) - f(x) \in \mathrm{co} \bigcup_{\varsigma \in (x,y)} \langle \underline{\partial} f(\varsigma) + \bar{\partial} f(\varsigma), y-x\rangle$.

Now we go tnto the case where $f: \mathbf{R}^n \to \mathbf{R}^m$ is a quasidifferentiable vector-valued function. To begin with, the following two properties of support functio recommended by Hiriart-Urruty (1980) are mentioned again, due to Hörmander (1954):

(P1) for any two nonempty closed convex sets A and B, $A \subset B$ if and only if $\delta_A^* \leq \delta_B^*$;

(P2) if $A = \bigcup_{i \in I} A_i$, then $\delta_{\overline{co}A}^* = \sup_{i \in I} \delta_{A_i}^*$, where $\delta_A^*(x^*) = \sup_{x \in A} \langle x, x^* \rangle$, $x \in X$, $x^* \in X^*$, and X is l.c.s., X^* is the dual space of X.

Given $x, y \in \mathbf{R}^n$, function f on cl (x, y) can be rewritten as $q(t) = f(x + t(y - x))$, $t \in$ cl $(0, 1)$. From Th. 1 and the proof of Hiriart-Urruty (1983), one has

$$\langle f(y) - f(x), z \rangle \leq \sup_{t \in (0,1)} \langle \underline{\partial} q(t) + \bar{\partial} q(t), z \rangle, \tag{5}$$

where $z \in \mathbf{R}^m$. It follows from (P2) that the inequality above can be rewritten as

$$\langle f(y) - f(x), z \rangle \leq \sup_{u \in M} \langle u, z \rangle,$$

where $M = \overline{co} \bigcup_{t \in (0,1)} (\underline{\partial} q(t) + \bar{\partial} q(t))$. According to (P1) one has

$$f(y) - f(x) \in \overline{co} \bigcup_{t \in (0,1)} (\underline{\partial} q(t) + \bar{\partial} q(t)). \tag{6}$$

Now it is necessary to calculate $Dq(t)$. It will be done below. First, the Jacobian matrix of a quasidifferentiable vector-valued function $f: \mathbf{R}^n \to \mathbf{R}^m$, $J_D f(s)$, $s \in \mathbf{R}^n$, is defined by

$$J_D f(s) = [\underline{J}_D f(s), \bar{J}_D f(s)],$$

where $\underline{J}_D f(s) = (\times_{k=1}^m \underline{\partial} f(s))^T$, $\bar{J}_D f(s) = (\times_{k=1}^m \bar{\partial} f(s))^T$, $f = (f_1, \cdots, f_m)^T$ and $f_i: \mathbf{R}^n \to \mathbf{R}$, $i = 1, \cdots, m$. $\underline{J}_D f$ and $\bar{J}_D f$ are called sub-Jacobian and super-Jacobian respectively. They are just the subdifferential and the superdifferential, respectively, when $m > 1$. Therefore

$$Df = \begin{cases} J_D f, & \text{if } m > 1, \\ J_D^{(T)} f, & \text{if } m = 1, \end{cases}$$

where $J_D^{(T)} f \triangleq [\underline{J}_D^T f, \bar{J}_D^T f]$.

According to the definition of quasidifferential, one has

$$f'(x + t(y - x); k(y - x)) = \max_{V \in \underline{J}_D f(x + t(y - x))} (Vk(y - x)) + \min_{W \in \bar{J}_D f(x + t(y - x))} (Wk(y - x))$$

$$= \max_{v \in \underline{J}_D f(x + t(y - x))(y - x)} kv + \min_{w \in \bar{J}_D f(x + t(y - x))(y - x)} kw,$$

where $k \in \mathbf{R}$. On the other hand

$$q'(t; k) = \max_{v \in \underline{J}_D q(t)} kv + \min_{w \in \bar{J}_D q(t)} kw.$$

Since it is easy to see that $q'(t; k) = f'(x + t(y - x); k(y - x))$, one can let

$$\underline{J}_D q(t) = \underline{J}_D f(x + t(y - x))(y - x), \tag{7}$$

and

$$\bar{J}_D q(t) = \bar{J}_D f(x + t(y - x))(y - x). \tag{8}$$

Finally, from (6), (7) and (8) the following holds

$$f(y) - f(x) \in \overline{co} \bigcup_{c \in (x, y)} (\underline{J}_D f(c) + \bar{J}_D f(c)) (y - x). \tag{9}$$

Furthermore there exist $\lambda_i \geq 0$, V_i, W_i and c_i, $i = 1, \cdots, m + 1$, such that

$$
\left.\begin{array}{l}
f(y)-f(x)=\sum_{i=1}^{m+1}\lambda_i(V_i+W_i)(y-x),\\
V_i\in \underline{\partial}_D f(c_i),\ W_i\in \bar{\partial}_D f(c_i),\\
\Sigma\lambda_i=1,
\end{array}\right\} \tag{10}
$$

given by Demyanov (1984). Thereby the following theorem has been proved.

Theorem 3 Suppose $f: \mathbf{R}^n\to\mathbf{R}^m$ is quasidifferentiable. Then for any $x, y\in \mathbf{R}^n$, (9) and (10) are true. ∎

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