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Periodic Solutions of Retarded Systems with Infinite Delay*

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§1 Introduction

It is well-known that the existence of the bounded solutions of linear periodic ordinary differential equations or functional differential equations with finite delay imply the existence of periodic solutions. In this paper, the author proves that the similar results are also true for the retarded systems with infinite delay in proper phase space; and also proves that the uniformly boundedness and uniformly ultimately boundedness of the solutions of the general retarded systems with infinite delay imply the existence of the periodic solutions. The results in [1] are extended.

§2 Phase Space

Let $\hat{X}$ be a linear vector space of functions mapping $R_- = (-\infty, 0]$ into R^n with elements designated by $\hat{\varphi}, \hat{\psi}, \cdots$, and $\hat{\varphi} = \hat{\psi}$ means $\hat{\varphi}(s) = \hat{\psi}(s)$ for all $s \leq 0$. The seminorm $|\cdot|_{\hat{X}}$ is given in $\hat{X}$. Assume that $X = \hat{X}/|\cdot|_{\hat{X}}$ is a Banach space with the norm $\|\cdot\|$ which is naturally induced by $|\cdot|_{\hat{X}}$. Elements of $\hat{X}$ are denoted by $\varphi, \psi, \cdots$, and correspond to equivalence classes of $\hat{X}$, for any element $\varphi \in X$, any element of the corresponding equivalence class will be denoted by $\hat{\varphi}$, and hence $\varphi = \psi$ in X means $\|\varphi - \psi\| = 0$.

For a real number $\beta \geq 0$ and a $\hat{\varphi} \in \hat{X}$, let $\hat{\varphi}^{\beta}$ denoted the restriction of $\hat{\varphi}$ to $(-\infty, -\beta]$, let $|\varphi|_{[-\beta,0]}$, $|\varphi|_{\beta}$ be the seminorm defined by

$$|\varphi|_{[-\beta,0]} = \inf_{\hat{\psi} \in \hat{X}} \left\{ \sup_{-\beta \leq \theta \leq 0} |\hat{\psi}(\theta)| : \psi = \varphi \right\}$$

$$|\varphi|_{\beta} = \inf_{\hat{\eta} \in \hat{X}} \left\{ \inf_{\hat{\psi} \in \hat{X}} \{ |\hat{\psi}|_{\hat{X}} : \hat{\psi}^{\beta} = \hat{\eta}^{\beta} \} : \eta = \varphi \right\}$$

X^{β} be the Banach space induced by $|\cdot|_{\beta}$. Define $\tau^{\beta}: X \to X^{\beta}$ as follows:

$$\tau^{\beta}\varphi = \{\varphi\}_{\beta} = \{\psi \in X : |\psi - \varphi|_{\beta} = 0\}$$

Assume that X satisfies the following axioms:

(H_1): For any $t_0 \in R$, $A \geq 0$, $x: (-\infty, t_0 + A] \to R^n$, $\varphi\ x_{t_0} = \hat{\varphi} \in \hat{X}$, $x(t)$ is

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continuous on $[t_0, t_0+A]$, then $x_t \in \hat{X}$ for all $t \in [t_0, t_0+A]$, and continuous in $t \in [t_0, t_0+A]$.

(H_2): There exists a continuous function $K(\beta)$ for $\beta \geq 0$ such that

$$\|\varphi\| \leq K(\beta)\,|\varphi|_{[-\beta,0]} + |\varphi|_\beta,\quad \varphi \in X.$$

(H_3): There is a locally bounded function $M(\beta)$ for $\beta \geq 0$ such that

$$|\tau^\beta \varphi|_\beta \leq M(\beta)\,\|\varphi\|,\quad \varphi \in X.$$

(H_4): There is a constant $K_0 > 0$ such that

$$|\varphi(0)| \leq K_0 \|\varphi\|.$$

Under the assumptions $(H_1 \sim H_4)$, it is easy to prove that the function in (H_1) satisfies the following inequality:

$$\|x_t\| \leq K(t-t_0) \sup_{t_0 \leq s \leq t} |x(s)| + M(t-t_0)\,\|x_{t_0}\|,\qquad t \in [t_0,\ t_0+A] \qquad (*)$$

§ 3 The Existence Theorem of The Periodic Solutions of Linear Systems

In this section, the following assumption for X is needed:

(Q_1): All the constant valued functions belong to X.

Consider the following retarded systems with infinite delay:

$$x'(t) = L(t,\ x_t) + h(t) \qquad (E_1)$$

where $L(t,\varphi)$ is continuous in $(t,\varphi) \in \mathbf{R} \times X$, linear in φ, $L(t+\omega,\varphi) = L(t,\varphi)$, $\omega =$ const>0; $h(t)$ is continuous on $\mathbf{R}$, $h(t+\omega) = h(t)$.

By the periodicity of $L(t,\varphi)$ in t and the linearity of $L(t,\varphi)$ in φ, it is obvious that the following results are true.

Lemma 1 There exists a constant $l > 0$ such that

$$|L(t,\varphi)| \leq l\,\|\varphi\| \qquad (t,\varphi) \in \mathbf{R} \times X$$

Lemma 2 Let $x(t)$ be the solution of (E_1) through (t_0, φ), then the following inequality is true:

$$\|x_t\| \leq \{K^*(t-t_0)\,\|\varphi\| + M^*(t,t_0)\}\,e^{l\int_{t_0}^{t} K(s-t_0)\,ds}$$

where

$$K^*(s) = \sup_{0 \leq \theta \leq s} [K_0 K(\theta) + M(\theta)],$$

$$M^*(s,t_0) = \sup_{t_0 \leq \theta \leq s} K(\theta - t_0) \int_{t_0}^{s} |h(\theta)|\,d\theta.$$

Definition 1 suppose a is the Kuratowski measure of noncompactness of a set in Banach space Z, and $f: Z \to Z$ is a continuous map. The map f is a-condensing if $a(fD) < a(D)$, for any bounded set $D \subset Z$ for which $a(D) > 0$ and fD is bounded; If there is a constant $\rho \in [0,1)$, such that $a(fD) < \rho a(D)$ for any bounded set $D \subset Z$ which fD is bounded, then the map f is said to be a a-contracting.

Now, let $X_0 = \{\varphi \in X, \varphi(0) = 0\}$. Define $S(t): X_0 \to X_0$ as follows:

$$[S(t)\varphi](\theta)=\begin{cases}0 & -t\le\theta\le 0\\ \varphi(t+\theta) & \theta<-t.\end{cases} \tag{1}$$

Then $S(t)$ is a bounded linear operator from X_0 to X_0.

The following theorem can be proved by lemma 1、2 and the similar way as theorem 5.2 in [2].

Theorem 3.1 Under the assumptions $(H_1\sim H_4)$ and (Q_1), the solution operator $T(t,t_0)$ of (E_1): $T(t,t_0)\varphi=x_t(t_0,\varphi)$, satisfies:

$$T(t,t_0)\varphi=U(t,t_0)\varphi+S(t-t_0)(\varphi-\varphi^*(0)),\qquad \varphi\in X$$

where $U(t,t_0)$ is a completely continuous operator, $\varphi^*(0)$ denote the function takes value $\varphi(0)$.

Lemma 3 [cf[1]], Let J: $Z\to Z$ be the linear bounded operator with spectrum contained in the open unit ball, then there is an equivalent norm $|||\cdot|||$ in Banach space Z such that $|||J|||<1$.

Lemma 4 (Darbo) Let D be a closed, bounded, convex subset of a Banach space Z and G: $D\to D$ is a-condensing, then G has a fixed point in D.

Now, we use the above lemmas to prove the following:

Theorem 3.2 suppose the spectral radius of the operator $S=S(\omega)$ is less than 1, then the existence of the bounded solutions of (E_1) implies the existence of ω-periodic solutions.

Proof Let $x(t_0,\varphi,h)(t)$ denote the solution of (E_1) through (t_0,φ), then

$$x(t_0,\varphi,h)(t)=x(t_0,\varphi,0)(t)+x(t_0,0,h)(t),\quad t\ge t_0. \tag{2}$$

Define the operator $T(t_0+\omega,t_0)$ as follows:

$$T(t_0+\omega,t_0)\varphi=x_{t_0+\omega}(t_0,\varphi,0)$$

By theorem 3.1, we know

$$T(t_0+\omega,t_0)\varphi=U(t_0+\omega,t_0)\varphi+S(\omega)\varphi, \tag{3}$$

where $U(t_0+\omega,t_0)$ is a completely continuous operator. The given condition in theorem 3.2 and the lemma 3 implies that there exists an equivalent norm $|||\cdot|||$ such that $|||S|||<1$, therefore, for any bounded set D of X

$$|||S\varphi|||\le|||S|||\,|||\varphi|||\quad \varphi\in D,$$

it implies

$$a(SD)\le|||S|||a(D)<a(D). \tag{4}$$

From (3) and (4) we have

$$a(T(t_0+\omega,t_0)D)=a(SD)<a(D). \tag{5}$$

That is, $T(t_0+\omega,t_0)$ is a-condensing.

Let x: $R\to R^n$ be a bounded solution of (E_1), $x_{t_0}=\varphi$; N be the set consists of all the positive integer numbers. Define the operator T_1: $T_1\varphi=x_{t_0+\omega}(t_0,\varphi,h)$, and the set D_0:

$D_0 = \left\{ \sum_{i \in P} a_i T_1^i \varphi \,\middle|\, a_i \geq 0,\ \sum_{i \in P} a_i = 1,\ P\text{; finite subset of } N \right\}$, then, by (2) and (5), we know that T_1 is α-condensing. It is easy to prove that the set $D = Cl(D_0)$ is a closed, bounded, convex subset of X and $T_1 D \subset D$. This complete the proof of the theorem by Lemma 4 and the uniqueness of the solution of (E_1) through (t_0, φ).

Example 1 For any $r>0$, et

$$\widetilde{C}_r = \{\widetilde{\varphi} \in C((-\infty, 0], R^n),\ e^{r\theta}\widetilde{\varphi}(\theta) \to \text{a limit as } \theta \to -\infty\}$$

and let

$$|\widetilde{\varphi}|_{C_r} = \sup_{-\infty<\theta\leq 0} |\widetilde{\varphi}(\theta)| e^{r\theta}$$

then by [2], we know that the space C_r induced by $|\cdot|_{C_r}$ satisfies the assumptions $(H_1 \sim H_4)$ and (Q_1), the spectral radius of $S(t)$ is less than $e^{rt}<1$, Therefore, by Theorem 3.2, if there is a bounded solution of (E_1), there must exists ω-periodic solution for (E_1).

We can prove the same result is also true for the space of Example 1.1 in [2] if the function g satisfies the inequality: $\sup_{\theta\leq 0} \frac{g(\theta - t)}{g(\theta)} < 1$, for $t>0$.

Remark If there is a positive integer m such that the spectral radius of $S(m\omega)$ is less than 1, then we can prove that the existence of bounded solutions of (E_1) implies the existence of $m\omega$-periodic solutions in the same way as above.

§ 4 The Periodic Solutions of General Retarded Systems

In order to discuss the general retarded systems, the following assumptions are supposed:

$(H_2)'$: The function $K(\beta)$ in (H_2) satisfies: $K(\beta) \leq K$;

$(H_3)'$: The function $M(\beta)$ in (H_3) satisfies: $\lim_{\beta \to +\infty} M(\beta) = 0$.

(H_5): If $\{\widetilde{\varphi}^n\}, \widetilde{\varphi}^n \in \hat{X}$, converges to $\widetilde{\varphi}$ uniformly on any compact set in R_- and $\{\varphi^n\}$ is a Cauchy sequence in X, then $\widetilde{\varphi} \in \hat{X}$, and $\lim_{n\to\infty} \|\varphi^n - \varphi\| = 0$.

Lemma 5 Under assumptions $(H_1 \sim H_5)$ and $(H_2)'$, $(H_3)'$, for any $L, c, c_1 > 0$, let

$W_c = \{\varphi \in X:\ \|\varphi\| \leq c\}$, $D^*(L, c_1, c) = \{x:\ R \to R^n, x_{t_0} \in W_c, x(t)$ is continuous for $t \geq t_0,\ \|x_t\| \leq c_1,\ |x(t) - x(s)| \leq L|t-s|, t, s \geq t_0\}$,

then the set

$$D(L, c_1, c) = \bigcap_{t \geq t_0} Cl\{x_s:\ x \in D^*(L, c_1, c), s \geq t\}$$

is a nonempty, convex, compact subset of X.

Proof The compactness of $D(L, c_1, c)$ is obtained by Theorem 3.1 in [2], obviously, $D(L, c_1, c)$ is a convex set in X, therefore, the result of Lemma 5 is true.

Consider the following retarded system with infinite delay:

$$x'(t) = F(t, x_t) \qquad (E_2)$$

where $F(t, \varphi)$ is continuous on $R \times X$, and if $x(t)$ is a solution of (E_2), then $x(t+\omega)$ is also a solution of (E_2). Furthermore, we assume that the solution $x(t_0, \varphi)$ of (E_2) through (t_0, φ) is existent and unique.

Definition 2 The solutions of (E_2) are said to be uniformly bounded, if for any $r>0$, there exists a positive number $B(r)$ such that $\|x_t(t_0, \varphi)\|<B(r)$ for all t_0, $\|\varphi\|\leq r, t\geq t_0$.

The solutions of (E_2) are said to be uniformly ultimately bounded, for $B>0$, if for any $r>0$, there is a $T(r)>0$ such that $\|x_t(t_0, \varphi)\|<B$, for all t_0, $\|\varphi\|\leq r, t\geq t_0+T(r)$.

Lemma 6 (Horn) Let $D_0\subset D_1\subset D_2$ be convex subsets of a Banach space Z, with D_0, D_2 be compact and D_1 open relative to D_2. Let $f: D_2\to Z$ be a continuous mapping such that, for some integer $m>0$

$$\bigcup_{i=1}^{m-1} f^i(D_1)\subset D_2, \qquad \bigcup_{i=m}^{2m-1} f^i(D_1)\subset D_0$$

then f has a fixed point on D_0.

Theorem 4.1 If the solutions of (E_2) are uniformly bounded and uniformly ultimately bounhed for $B>0$, $F(t, \varphi)$ takes $R\times G$ ($G\subset X$, bounded and closed) into bounded set, then there exists a ω-periodic solution of (E_2).

Proof For $B>0$, by the uniformly boundedness of the solutions, there is a $B_1>0$ such that, $\|x_t(t_0, \varphi)\|<B_1$, for all $\|\varphi\|\leq B, t\geq t_0$, there also exists $B_2\geq B_1+1$ such that, $\|\varphi\|\leq B_1+1$ implies $\|x_t(t_0, \varphi)\|<B_2, t\geq t_0$; by the uniformly ultimately boundedness of the solutions, there exists a positive integer m such that

$$\|x_t(t_0, \varphi)\|<B, \text{ if } \|\varphi\|\leq B_2, \quad t\geq t_0+(m-1)\omega$$

there is a constant $L>0$ such that $|F(t, \varphi)|\leq L$, for all $(t, \varphi)\in R\times W_{B_2}$, since $F(t, \varphi)$ takes $R\times W_{B_2}$ into bounded set.

Let
$$D_0 = D(L, B, B), \quad D_2 = D(L, B_1+1, B_2),$$
$$D_1 = \{\varphi\in X;\ \|\varphi\|\leq B_1+1\}\cap D_2,$$

then D_1 is open relative to D_2, $D_0\subset D_1\subset D_2$, Lemma 5 implies that, D_0, D_2 are nonempty, convex, compact subsets of X. Define the operator $f: D_2\to X$ as follows:

$$f\varphi = x_\omega(0, \varphi) \qquad \varphi\in D_2.$$

From the following inequalities:

$$\|x_t(0, \varphi)\|<B_1, t\geq 0, \quad \|\varphi\|\leq B,$$
$$\|x_t(0, \varphi)\|<B_2, t\geq 0, \quad \|\varphi\|\leq B_1+1,$$

$$\|x_t(0,\varphi)\|<B,\ t\geq(m-1)\omega,\quad \|\varphi\|\leq B_2,$$

and the definition of the set $D(L,c_1,c)$, we have

$$\bigcup_{i=1}^{m-1} f^i(D_1)\subset D_2,\quad \bigcup_{i=1}^{2m-1} f^i(D_1)\subset D_0.$$

Lemma 6 implies that there exists a φ D_0 such that $f\varphi=\varphi$, i.e.: $x_\omega(0,\varphi)=\varphi$, by the uniqueness of the solution of (E_2) through $(0,\varphi)$, we know that $x(0,\varphi)(t)$ is a ω-periodic solution of (E_2) for $t\geq 0$.

Theorem 4.2 Under assumptions $(H_1\sim H_4)$, if there exists a nonzero solution $\bar{x}(t_0,\bar{\varphi})(t)$ of (E_2), and positive integer m, $\varphi_1\in X$, $(\varphi_1\neq 0)$, such that

$$\lim_{n\to\infty}\|\bar{x}_{nm\omega+t_0}(t_0,\bar{\varphi})-\varphi_1\|=0$$

then there exists a $m\omega$-periodic solution of (E_2).

Proof By the uniqueness and continuous dependence on initial functions of the solutions, we get

$$\begin{aligned}x(t_0,\varphi_1)(t)&=x(t_0,\lim_{n\to\infty}\bar{x}_{nm\omega+t_0}(t_0,\bar{\varphi}))\ (t)=\lim_{n\to\infty}x(t_0,\bar{x}_{nm\omega+t_0}(t_0,\bar{\varphi}))\ (t)\\&=\lim_{n\to\infty}x(t_0,\bar{\varphi})(t+nm\omega)=\lim_{n\to\infty}x(t_0,\bar{x}_{(n-1)m\omega+t_0}(t_0,\bar{\varphi}))(t+m\omega)\\&=x(t_0,\lim_{n\to\infty}\bar{x}_{(n-1)m\omega+t_0}(t_0,\bar{\varphi}))(t+m\omega)=x(t_0,\varphi_1)(t+m\omega)\end{aligned}$$

that is, $x(t_0,\varphi_1)(t)$ is a $m\omega$-periodic solution as desired.

We can study the boundedness of the solutions of (E_2) by using the Liapunov's direct method in the similar way as functional differential equations with finite delay, such results are omitted here.

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