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A Note on A Coupled Reaction-diffusion System with Time Delays*

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Abstract The global stability of the steady-state solution to a coupled reaction-diffusion system with time delays is investigated. All factors involved in the system, especially, the permeabilites and diffusion are taken into account, and the relevant results of the papers [1, 2] are thus improved.

Some biochemical reaction can be modeled by a coupled system of time-delayed ordinary differential equations and linear parabolic partial differential equations like this (cf. [1, 2]):

$$\begin{cases} u_1' + (a_1+b_1)u_1 = a_1u_2(0,\ t) + f(v_1(t-r_1)), \\ v_1' + (a_1+b_2)v_1 = a_1v_2(0,\ t), \\ (u_2)_t - D_1(u_2)_{xx} + b_1u_2 = 0, \\ (v_2)_t - D_2(v_2)_{xx} + b_2v_2 = 0, \\ u_3' + (a_2+b_1)u_3 = a_2u_2(l,\ t), \\ v_3' + (a_2+b_2)v_3 = a_2v_2(l,\ t) + g(u_3(t-r_2)), \end{cases} \quad 0<x<l,\ t>0 \tag{1}$$

where $u_1' = \mathrm{d}u_1/\mathrm{d}t$, $(u_2)_t = \partial u_2/\partial t$, $(u_2)_{xx} = \partial^2u_2/\partial x^2$, etc., a_i, b_i, D_i, r_i are positive constants corresponding to the membrane permeabilities, the reaction rates, the diffusion coefficients, the time delays respectively, and f, g are given reaction functions. The boundary conditions for u_2 and v_2 are

$$\begin{cases} -(u_2)_x(0,\ t) + \beta_1u_2(0,\ t) = \beta_1u_1(t), \\ (u_2)_x(l,\ t) + \beta_2u_2(l,\ t) = \beta_2u_3(t), \\ -(v_2)_x(0,\ t) + \beta_1^*v_2(0,\ t) = \beta_1^*v_1(t), \\ (v_2)_x(l,\ t) + \beta_2^*v_2(l,\ t) = \beta_2^*v_3(t), \end{cases} \quad t>0, \tag{2}$$

where β_i and β_i^* are positive constants associated with the permeabilities. The initial conditions for the system are given by

$$\begin{cases} u_1(0) = \xi_1, & v_1(t) = \eta_1(t), & -r_1 \le t \le 0, \\ u_2(x,\ 0) = \xi_2(x), & v_2(x,\ 0) = \eta_2(x), & 0<x<l, \\ u_3(t) = \xi_3(t), & v_3(0) = \eta_3, & -r_2 \le t \le 0, \end{cases} \tag{3}$$

where ξ_1, η_3 are nonnegative constants and ξ_2, ξ_3, η_1, η_2 are given continuous nonnegative functions of their respective arguments. Assuming

$$f(v) \ge 0,\quad f'(v) \le 0 \text{ for } v \ge 0 \tag{4}$$

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and

$$g(u) = c_2 u, \qquad c_2 > 0,$$

J. M. Mahaffy and C. V. Pao [1, 2] have given a qualitative analysis of the problem (1)—(3). They proved the existence and uniqueness of solutions to this problem and to the corresponding steady-state problem. Under the additional restriction

$$0 \le -f'(v) \le c_1 < b_1 b_2 / c_2 \quad \text{for} \quad v \ge 0, \tag{5}$$

they also proved the globel stability of the steady-state solution.

It is clear that the restriction (5) fails to represent the effects of the parameters other than b_1, b_2 and c_2. The purpose of this note is to take into account the effects of all parameters and improve the stability result mentioned above.

Assuming (4) and

$$g(u) \ge 0, \quad g'(u) \ge 0 \quad \text{for} \quad u \ge 0, \tag{6}$$

and using the argument similar to that used in [1, 2], we can prove that the problem (1)—(3) has an unique solution (u_i, v_i) and an unique steady-state solution (u_i^s, v_i^s). Set

$$c_1 = -\inf_{v \ge 0} f'(v), \quad c_2 = \sup_{u \ge 0} g'(u).$$

The main result of this note is

Theorem Suppose that (4) and (6) hold and that

$$c_1 c_2 < K(a_1 + b_2)(a_2 + b_1) b_1 b_2 / a_1 a_2 \tag{7}$$

where $K = K(a_i, b_i, D_i, \beta_i, \beta_i^*) > 1$. Then for any initial data (ξ_i, η_i), $i = 1, 2, 3$, there exist positive constants M and ε independent of the time delays r_1 and r_2 such that

$$\begin{cases} |u_i(t) - u_i^s|, \quad |v_i(t) - v_i^s| \le M e^{-\varepsilon t}, \quad i = 1, 3, \\ |u_2(x, t) - u_2^s(x)|, \quad |v_2(x, t) - v_2^s(x)| \le M e^{-\varepsilon t} \end{cases} \qquad 0 \le x \le l, \ t \ge 0. \tag{8}$$

Proof It is easy to see that the differences $\hat{u}_i = u_i - u_i^s$ and $\hat{v}_i = v_i - v_i^s$ satisfy

$$\begin{cases} \hat{u}_1' + (a_1 + b_1)\hat{u}_1 = a_1 \hat{u}_2(0, t) + \int_0^1 f'(\tau v_1(t - r_1) + (1 - \tau) v_1^s) d\tau \hat{v}_1(t - r_1), \\ \hat{v}_1' + (a_1 + b_2)\hat{v}_1 = a_1 \hat{v}_2(0, t), \\ (\hat{u}_2)_t - D_1(\hat{u}_2)_{xx} + b_1 \hat{u}_2 = 0, \\ (\hat{v}_2)_t - D_2(\hat{v}_2)_{xx} + b_2 \hat{v}_2 = 0, \qquad\qquad 0 < x < l, \quad t > 0, \\ \hat{u}_3' + (a_2 + b_1)\hat{u}_3 = a_2 \hat{u}_2(l, t), \\ \hat{v}_3' + (a_2 + b_2)\hat{v}_3 = a_2 \hat{v}_2(l, t) + \int_0^1 g'(\tau u_3(t - r_2) + (1 - \tau) u_3^s) d\tau \hat{u}_3(t - r_2), \end{cases} \tag{9}$$

$$\begin{cases} -(\hat{u}_2)_x(0, t) + \beta_1 \hat{u}_2(0, t) = \beta_1 \hat{u}_1(t), \\ (\hat{u}_2)_x(l, t) + \beta_2 \hat{u}_2(l, t) = \beta_2 \hat{u}_3(t), \\ -(\hat{v}_2)_x(0, t) + \beta_1^* \hat{v}_2(0, t) = \beta_1^* \hat{v}_1(t), \\ (\hat{v}_2)_x(l, t) + \beta_2^* \hat{v}_2(l, l) = \beta_2^* \hat{v}_3(t), \end{cases} \quad t > 0 \tag{10}$$

and

$$\begin{cases} \hat{u}_1(0) = \xi_1 - u_1^s, & \hat{v}_1(t) = \eta_1(t) - v_1^s, \quad -r_1 \le t \le 0, \\ \hat{u}_2(x, 0) = \xi_2(x) - u_2^s(x), & \hat{v}_2(x, 0) = \eta_2(x) - v_2^s(x), \quad 0 < x < l, \\ \hat{u}(t) = \xi_3(t) - u_3^s & \hat{v}_3(0) = \eta_3 - v_3^s, \quad -r_2 \le t \le 0. \end{cases} \tag{11}$$

Set

$$u_i^{\pm}(t) = \pm p_i e^{-\varepsilon t}, \quad v_i^{\pm}(t) = \pm q_i e^{-\varepsilon t}, \quad i = 1, 3, \tag{12}$$

$$u_2^{\pm}(x, t) = \pm p_2 \varphi(x) e^{-\varepsilon t}, \quad v_2^{\pm}(x, t) = q_2 \psi(x) e^{-\varepsilon t} \quad 0 \le x \le l, \ t \ge 0, \tag{13}$$

where $\varepsilon < \min(b_1, b_2)$, p_i, q_i, $i = 1, 2, 3$, are positive constants and φ, ψ are positive functions to be determined later. According to the monotone method (see [2]), in order to prove (8) it suffices to show that (u_i^+, v_i^+) and (u_i^-, v_i^-) given by (12) and (13) are the upper and lower solutions of the problem (9)—(11), respectively. In other words, it suffices to show (u_i^+, v_i^+) and (u_i, v_i^-) satisfy (9)—(11) with "=" replaced by "$\ge$" and "$\le$" respectively. It turns out that we need only to select p_i, q_i, $\varphi(x)$ and $\psi(x)$ such that

$$\begin{cases} (a_1 + b_1 - \varepsilon) p_1 \ge a_1 \varphi(0) p_2 + c_1 e^{\varepsilon r_1} q_1 \\ (a_1 + b_2 - \varepsilon) q_1 \ge a_1 \psi(0) q_2, \\ [(b_1 - \varepsilon)\varphi(x) - D_1 \varphi''(x)] p_2 \ge 0, \\ [(b_2 - \varepsilon)\psi(x) - D_2 \psi''(x)] q_2 \ge 0, \quad 0 < x < l, \\ (a_2 + b_1 - \varepsilon) p_3 \ge a_2 \varphi(l) p_2, \\ (a_2 + b_2 - \varepsilon) q_3 \ge a_2 \psi(l) q_2 + c_2 e^{\varepsilon r_2} p_3, \end{cases} \tag{14}$$

$$\begin{cases} [\beta_1 \varphi(0) - \varphi'(0)] p_2 \ge \beta_1 p_1, \\ [\beta_2 \varphi(l) + \varphi'(l)] p_2 \ge \beta_2 p_3, \\ [\beta_1^* \psi(0) - \psi'(0)] q_2 \ge \beta_1^* q_1, \\ [\beta_2^* \psi(l) + \psi'(l)] q_2 \ge \beta_2^* q_3 \end{cases} \tag{15}$$

and

$$\begin{cases} p_1 \ge |\xi_1 - u_1^s|, & q_1 \ge |\eta_1(t) - v_1^s|, \quad -r_1 \le t \le 0, \\ p_2 \varphi(x) \ge |\xi_2(x) - u_2^s(x)|, & q_2 \psi(x) \ge |\eta_2(x) - v_2^s|, \quad 0 < x < l, \\ p_3 \ge |\xi_3(t) - u_3^s|, & q_3 \ge |\eta_3 - v_3^s|, \quad -r_2 \le t \le 0. \end{cases} \tag{16}$$

Before going on we point out the following fact.

Lemma $\forall$ positive a, b, and β, $\exists$ a positive function $\varphi = \varphi(x; a, b, D, \beta)$ such that

$$b\varphi(x) - D\varphi''(x) \ge 0 \quad \text{for } 0 < x < l,$$

$$\varphi(0) = \varphi(l) = 1,$$

$$\varphi'(0) < 0 \quad \text{and} \quad \varphi'(l) \ge -\frac{b\beta}{a + b}.$$

Actually, there are many such functions. For example,

$$\varphi_1(x) = 1 - \lambda x(l - x), \qquad 0 < \lambda \le 4b/(8D + bl^2),$$

$$\varphi_2(x)=e^{-\lambda x}(l-x), \qquad 0<\lambda\le\left(\sqrt{1+\frac{bl^2}{D}}-1\right)/l^2,$$

$$\varphi_3(x)=1-\lambda\sin(2\pi x/l), \quad 0<\lambda\le\min\left\{\frac{b}{b+4\pi^2/l^2D},\ \frac{b\beta l}{2\pi(a+b)}\right\},$$

$$\varphi_4(x)=e^{-\lambda\sin(\pi x/l)}, \qquad 0<\lambda\le\begin{cases}\left(\frac{l}{\pi}\right)^2\frac{b}{D} & \text{if } \frac{b}{D}\le\frac{1}{2},\\ \left(\frac{l}{\pi}\right)^2\sqrt{\frac{b}{D}-\frac{1}{4}} & \text{if } \frac{b}{D}>\frac{1}{2},\end{cases}$$

$$\varphi_5(x)=\operatorname{ch}(\lambda x)-2\frac{\operatorname{sh}^2(\lambda l/2)}{\operatorname{sh}(\lambda l)}\operatorname{sh}(\lambda x), \quad 0<\lambda\le\sqrt{b/D},$$

etc. Now we choose

$$\varphi(x)=\varphi_\varepsilon(x)=\varphi(x;\ a_2,\ b_1-\varepsilon,\ D_1,\ \beta_2),$$
$$\psi(x)=\psi_\varepsilon(x)=\varphi(l-x;\ a_1,\ b_2-\varepsilon,\ D_2,\ \beta_1^*).$$

$$p_2/p_1=\beta_1/[\beta_1-\varphi'_\varepsilon(0)], \tag{17}$$

$$p_3/p_2=a_2/(a_2+b_1-\varepsilon), \tag{18}$$

$$p_1/q_2=a_1/(a_1+b_2-\varepsilon), \tag{19}$$

and

$$q_2/q_3=\beta_2^*/[\beta_2^*+\psi'_\varepsilon(l)]. \tag{20}$$

Then all the inequalities in (14) except for the first and the last ones, and (15) are fulfilled. In order that these two inequalities also hold we need

$$\left[b_1-\varepsilon-\frac{\varphi'_\varepsilon(0)}{\beta_1}(a_1+b_1-\varepsilon)\right]p_2\ge c_1e^{\varepsilon r_1}\frac{a_1}{a_1+b_2-\varepsilon}q_2$$

and

$$\left[b_2-\varepsilon-\frac{\psi'_\varepsilon(l)}{\beta_2^*}(a_2+b_2-\varepsilon)\right]q_2\ge c_2e^{\varepsilon r_2}\frac{a_2}{a_2+b_1-\varepsilon}p_2,$$

i.e.,

$$\frac{a_1c_1\beta_1e^{\varepsilon r_1}}{(a_1+b_2-\varepsilon)[(b_1-\varepsilon)\beta_1-\varphi'_\varepsilon(0)(a_1+b_1-\varepsilon)]}\le\frac{p_2}{q_2}\le\frac{(a_2+b_1-\varepsilon)[(b_2-\varepsilon)\beta_2^*+\psi'_\varepsilon(l)(a_2+b_2-\varepsilon)]}{a_2c_2\beta_2^*e^{\varepsilon r_2}}. \tag{21}$$

This is possible if

$$\frac{a_1c_1\beta_1}{(a_1+b_2)[b_1\beta_1-\varphi'_0(0)(a_1+b_1)]}<\frac{(a_2+b_1)[b_2\beta_2^*+\psi'_0(l)(a_2+b_2)]}{a_2c_2\beta_2^*} \tag{22}$$

and ε sufficiently small. The inequality (22) is nothing but (7) with

$$K=\frac{[b_1\beta_1-\varphi'_0(0)(a_1+b_1)][b_2\beta_2^*+\psi'_0(l)(a_2+b_2)]}{b_1t_2\beta_1\beta_2^*}>1. \tag{23}$$

By taking p_i and q_i big enough and remaining (17)—(21) unchanged, (16) can also be satisfied, and the theorem is thus proved.

Remark From (7), (23) and the examples for $\varphi(x)$ we see that not only small c_1, c_2 and big b_1, b_2, as shown in (5), but also small a_1, a_2, D_1, , D_2, β_1, β_2^* and big β_1^*, β_2 can ensure the global stability.

References

[1] J. M. Mahaffy and C. V. Pao, Models of genetic control by repression with time delays and spatial effects, J. Math. Biology, 20(1984), 39—57.

[2] C. V. Pao and J. M. Mahaffy, Qualitative analysis of a coupled reaction-diffusion model in biology with time delays, J. Math. Anal. Appl., 109 (1985), 355—371.

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integers n, m, m' such that $b^2 = na^2$, $ab = ma^2$, $ba = m'a^2$ and for which the congruence equation $X^2 + (m+m')X + n \equiv 0 \pmod{q_j}$ has no integer solution X.

(5) $\mathbf{R} = \mathbf{P} \oplus \mathbf{K}$ where $\mathbf{P} = \sum_{p_i \in \mathbf{I}} \oplus \mathbf{F}_{p_i}$ satisfies (1) and $\mathbf{K} = \sum_{q_i \in \mathbf{J}} \mathbf{R}_{q_j}$ satisfies (4).

(6) $\mathbf{R} = \mathbf{K} + \mathbf{R}_1$ where $\mathbf{K} = \sum_{q_i \in \mathbf{J}} \mathbf{R}_{q_j}$ satisfies (4) and $\mathbf{R}_1 = \{r_0\}$, $o(r_0) = \infty$, $r_0^2 = nr_0$ for $n \neq 0$. For each element x of $\mathbf{K}$, $xr_0 = 0$, $r_0 x = nx$.

(7) $\mathbf{R} = \mathbf{K} + \mathbf{A}$, where $\mathbf{K} = \sum_{q_i \in \mathbf{J}} \mathbf{R}_{q_j}$ satisfies (4) and $\mathbf{A} = \mathbf{R}_1 \oplus \sum_{p_i \in \mathbf{I}} \oplus \mathbf{F}_{p_i}$ satisfies (3), $\mathbf{R}_1 = \{r_0\}$, $o(r_0) = \infty$, $r_0^2 = nr_0$. For each $x \in \mathbf{K}$, $r_0 x = nx$, and for each $x \in \mathbf{K}$, $a \in \mathbf{A}$, $b \in \sum_{p_i \in \mathbf{I}} \oplus \mathbf{F}_{p_i}$, $xa = 0$, $bx = 0$.

Reference

[1] Liu Shaoxue, J.Beijing Normal Univ. (natural sci.), 1979, no. 3, 1—6.