J. Math. Res. & Exposition, 1988, No. 2 .

A Note on Asymptotic Joint Distribution of The Eigenvalues of A Noncentral Multivariate F Matrix

Z. D. Bai

China University of Science and Technology
and
Center for Multivariate Analysis, University of Pittsburgh

Abstract In P. L. Hsu (1941), the proof of the basic Lemma 3 is based on Lemma 1 which is wrong. The aim of this note is to correct the proof of Lemma 3, consequently, to ensure the main theorem in P. L. Hsu (1941).

I. Introduction

Consider k p-variate normal populations with mean vectors $(\xi_{1i}, \xi_{2i}, \cdots, \xi_{pi})$, $i = 1, 2, \cdots, k$ and a common covariance matrix $\Sigma = \|\sigma_{ij}\|$. Now, let $\xi_i' = (\xi_{1i}, \xi_{2i}, \cdots, \xi_{pi})$ and

$$\xi = \begin{bmatrix} \xi_{11}, & \xi_{21}, \cdots, & \xi_{p1}, & 1 \\ \xi_{12}, & \xi_{22}, \cdots, & \xi_{p2}, & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{1k}, & \xi_{2k}, \cdots, & \xi_{pk}, & 1 \end{bmatrix}. \tag{1}$$

The geometrical meaning of the rank of ξ is obvious, if its rank is one, the k centroids of the k populations are coincident, if it is two, the centroids are collinear but not coincident, and so on. So, the rank of ξ is important in certain problems of inference in the area of multivariate analysis.

Suppose there are k samples, with size m_1, $m_2 \cdots$, m_k, respectively, drawn from the k populations. Let

$$\begin{cases} \xi_{i\cdot} = \dfrac{1}{N}\sum_{t=1}^{k} m_t \xi_{it} \\ \psi_{ij} = \dfrac{1}{N}\sum_{t=1}^{k} m_t (\xi_{it} - \xi_{i\cdot})(\xi_{jt} - \xi_{j\cdot}) \\ N = \sum_{i=1}^{k} m_i \end{cases} \tag{2}$$

* *Received* *Mar.* 20, 1985.
Key Words: Limiting distribution, multivariate F matrix, eigenvalues of random matrix.
Technical Report No. 84-49, *Nov.*, 1984.

It is not difficult to see that the rank of the matrix $\|\psi_{ij}\|$ is one less than that of ξ. On the other hand, the rank of $\|\psi_{ij}\|$ is in turn equal to the number of positive roots of the determinantal equation

$$|\psi_{ij} - \lambda\sigma_{ij}| = 0 \qquad (3)$$

because the matrix $\|\psi_{ij}\|$ is nonnegative definite and $\Sigma = \|\sigma_{ij}\|$ is positive definite. Thus, the problem to investigate the rank of ξ turns out to be the problem to investigate the number of positive roots of (3). A review of the literature on testing for the rank of $\Sigma^{-1}\psi$ is given in Krishnaiah (1982).

Now, let us consider the samples drawn from the k multivariate normal populations. Let $(x_{1t}, \cdots, x_{pt})$ denote the mean vector of the t-th sample and $(x_1, \cdots, x_p)$ denote the mean vector of the grand sample, s_{ijt} (i, $j = 1, 2, \cdots, p$) the second moments about the means of the t-th sample. Write

$$\overline{a_{ij}} = \sum_{t=1}^{k} m_t'(x_{it} - x_i)(x_{jt} - x_j)$$

$$\overline{b_{ij}} = \sum_{t=1}^{k} m_t s_{ijt}$$

$$l_2 = \min(p, \ k-1)$$

$$l_2 = \max(p, \ k-1) \ .$$

It is obvious that the matrix $\|\overline{a_{ij}}\|$ is nonnegative definite and has rank l_1, and that the matrix $\|\overline{b_{ij}}\|$ is positive definite provided that $N - k \geqslant p$. Hence the determinantal equation in ϕ

$$|\overline{a_{ij}} - \phi\overline{b_{ij}}| = 0 \qquad (4)$$

has a root zero of multiplicity $p - l_1$ and l_1 positive roots.

The l_1 nonzero roots of (4) play an important role in discriminant analysis. Their distribution depends solely upon the roots of (3), and their exact distribution is known in the case that where all roots of (3) are zero (see Fisher (1939), Hsu (1939), and Roy (1939)). In the general case, Hsu (1941) obtained the limiting distribution of these positive roots and it is given in the following theorem.

Theorem (P. L. Hsu). Suppose the sample sizes satisfy the condition $m_t = mq_t$, $t = 1, 2, \cdots, k$. Let $q = \sum_{t=1}^{k} q_t$ and $N = mq$ (note that the roots of (3) is independent of m in the present case). Suppose that (3) has positive roots $\lambda_1 > \lambda_2 > \cdots > \lambda_\nu > 0$ with multiplicities $\mu_1, \mu_2, \cdots, \mu_\nu$, respectively. Write

$$a_0 = 0, \ a_h = a_{h-1} + \mu_h, \ h = 1, 2, \cdots, \nu, \ r = a_\nu \ .$$

Also write the positive roots of (4) as $\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_{l_1} > 0$. Define

$$\zeta_i = \sqrt{N}\,(2\lambda_h^2 + 4\lambda_h)^{-\frac{1}{2}}(\phi_i - \lambda_h), \quad (i = a_{h-1} + 1, \cdots, a_h, \ h = 1, 2, \cdots, \nu)$$

$$\zeta_i = N\phi_i, \qquad (i = r+1, \cdots, l_1)$$

Then the limiting distribution density (as $m \to \infty$) of $\zeta_1, \cdots, \zeta_{l_1}$ is given by

$$D(\zeta_1, \cdots, \zeta_{a_1}) D(\zeta_{a_1+1}, \cdots, \zeta_{a_2}) \cdots D(\zeta_{a_{v-1}+1}, \cdots, \zeta_{a_v}) D_1(\zeta_{r+1}, \cdots, \zeta_{l_1}) \tag{5}$$

where

$$D(x_1 \cdots x_\mu) = \left(\frac{1}{2}\right)^{\frac{1}{2}\mu} \left(\prod_{i=1}^{\mu} \Gamma\left(\frac{i}{2}\right)\right)^{-1} \left\{\prod_{i=1}^{\mu} \prod_{j=i+1}^{\mu} (x_i - x_j)\right\} \cdot$$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{\mu} x_i^2\right\} \quad (\infty > x_1 \geq x_2 \geq \cdots \geq x_\mu > -\infty) \tag{6}$$

$$D_1(\zeta_{r+1}, \cdots, \zeta_{l_1}) = \left(\frac{1}{2}\right)^{\frac{1}{2}(p-r)(n_1-r)} \pi^{\frac{1}{2}(l_1-1)} \left\{\prod_{i=1}^{l_1-1} \Gamma\left(\frac{1}{2}l_2 - \frac{1}{2}r - \frac{i}{2} + \frac{1}{2}\right) \Gamma\left(\frac{i}{2}\right)\right\}^{-1} \cdot$$

$$\left\{\prod_{i=r+1}^{l_1} \prod_{j=r+1}^{l_1} (\zeta_i - \zeta_j)\right\} \left\{\prod_{i=r+1}^{l_1} \zeta_i\right\}^{\frac{1}{2}(l_2 - l_1 - 1)} \exp\left\{-\frac{1}{2}\sum_{i=r+1}^{l_1} \zeta_i\right\} \quad (\infty > \zeta_{r+1} \geq \cdots \geq$$

$$\zeta_{l_1} \geq 0, \quad n_1 = k-1). \tag{7}$$

Unfortunately, Lemma 1 in the paper of Hsu (1941) is not correct. The main result of Hsu is based upon the above lemma. Recently, Prof. W. Q. Liang found this mistake and gave a counterexample. The purpose of this note is, according to the suggestion of Professor K. L. Chung, to give a new proof of Lemma 3 in Hsu's paper; this lemma plays a key role in the proof of the main theorem of the result of Hsu.

2. A Counterexample

To clearly understand the counterexample, we have to restate here the Lemma 1 in Hsu (1941).

Let $Q_n (n = 1, 2, \cdots)$ be a random point with a finite number, independent of n, of coordinates, and let its domain of existence be the Borel set $\mathbf{E}_n$. Let

$$\mathbf{E}_n \subset \mathbf{E}_{n+1} \ (n = 1, 2, \cdots)$$

and put

$$\lim_{n \to \infty} \mathbf{E}_n = \mathbf{E}.$$

Let the probability of Q_n approach a limiting probability function, which is continuous, as $n \to \infty$.

Let $f_n(P)(n=1,2,\cdots)$ be a real point function defined and Borel measurable in E_n. Let

$$\lim_{n\to\infty} f_n(P)=0 \text{ throughout } E$$

then the random variable $f_n(Q_n)\to 0$ in pr. as $n\to\infty$.

Example 1. Let p_n denote the n^{th} prime number, and let $F_n=\{\frac{1}{p_n},\frac{2}{p_n},\cdots,\frac{p_n-1}{p_n}\}$. Define

$$f_n(P)=\begin{cases}1 & \text{if } p\in F_n\\ 0 & \text{otherwise.}\end{cases}$$

Evidently, we have

$$f_n(P)\to 0,\ \forall\ p\in(0,\ 1).$$

Define random variable Q_n with its distribution function

$$F_n(x)=P(Q_n<x)=\frac{k_n(x)}{2p_n}+\frac{p_n+1}{2p_n}\int_{-\infty}^{x} I_{(0,1)}(u)\,du$$

where $k_n(x)$ is the number of elements of F_n, which are less than x, and I_A denotes the indicator function of the set A. It is obvious that F_n tends to $R(0,\ 1)$, the uniform distribution over the interval $(0,\ 1)$, as $n\to\infty$. Also, we know that $E=E_n=(0,\ 1)$, $n=1,2,\cdots$. But we evidently have

$$P(f_n(Q_n)=1)=P(Q_n\in F_n)=\frac{p_n-1}{2p_n}\to\frac{1}{2}.$$

In fact, we can construct an example in which all the $f_n(P)$, $n=1,2,\cdots$, are continuous and all the distributions of Q_n, $n=1,2,\cdots$, are absolutely continuous, but the conclusion of this Lemma is not true. On the other hand, we can show that if each Q_n has a probability density q_n and q_n tends to a limit density q, then the conclusion of this lemma will be true.

But, we omit the details since the main purpose of our paper is to give a correct proof of Hsu's theorem and not to give details of counterexamples of his lemma.

3. Proof of Hsu's Theorem

write

$$n_1 = k-1, \quad n = N-k, \quad v = N^{-\frac{1}{2}}$$

$$a_{ij} = \sum y_{i\beta} y_{j\beta}, \quad u_{ii} = \frac{1}{\sqrt{2N}} \left(\sum_{v=1}^{n} z_{iv}^2 - N \right),$$

$$u_{ij} = \frac{1}{\sqrt{N}} \sum_{v=1}^{n} z_{iv} z_{jv} \quad i \neq j, \quad 1 \leqslant i, \quad j \leqslant p$$

In [5], it is correctly proved that the distribution of the positive roots of (4) is the same as that of the following determinantal equation

$$\left| v^2 A + vC - v\phi U + D \right| = 0 \tag{9}$$

where

$$A = \| a_{ij} \|, \quad U = \left\| \begin{matrix} \sqrt{2}u_{11}, & \cdots, & u_{1p} \\ \vdots & & \vdots \\ u_{p1}, & \cdots, & \sqrt{2}u_{pp} \end{matrix} \right\|$$

$$D = \left\| \begin{matrix} (\lambda - \phi) I_{\mu} & & & \\ & 1 \ddots & & \\ & & (\lambda_v - \phi) I_{\mu_v} & \\ & & & -\phi I_{p-r} \end{matrix} \right\|$$

$$C = \begin{bmatrix} C_{11} & \cdots C_{1v} & E_v' \\ \vdots & \vdots & \vdots \\ C_{1v}' & \cdots C_{vv} & E_v' \\ E_1 & \cdots E_v & 0 \end{bmatrix}$$

$C_{hh} = \sqrt{\lambda_h} \| y_{ij} + y_{ji} \|$, i, $j = a_{h-1}+1, \cdots, a_h$, $h = 1, 2, \cdots,$.

$C_{hg} = \| \sqrt{\lambda_h}\, y_{jl} + \sqrt{\lambda_g}\, y_{ij} \|$, $h < g$, $l = a_{h-1}+1, \cdots, a_h$, $j = a_{g-1}+1, \cdots,$ a_g,

$E_h = \sqrt{\lambda_h} \| y_{ij} \|$, $i = r+1, \cdots, p$, $j = a_{h-1}+1, \cdots, a_h$,

and $\{y_{ij}, z_{jv}\}$ has joint probability density

$$(2\pi)^{-\frac{1}{2}p(n_1+n)} \exp\left\{ -\frac{1}{2}\left(\sum_{l=1}^{p} \sum_{j=1}^{n_1} y_{l}^2 + \sum_{l=1}^{p} \sum_{v=1}^{n} z_{lv}^2 \right\} \tag{10}$$

In Lemma 2 of [5], it is shown that all the u_{ij}'s tend to iid.

$N(0, 1)$'s. Though Lemma 1, which is not true, was used in the proof of Lemma 2, the correctness of Lemma 2 is easily seen. Its proof can be easily modified and is omitted. The reader can refer to Lemmas 9 and 10 in X. R. Chen (1981).

According to this lemma, the u_{ij}'s in (9) was replaced by a set of iid. $N(0, 1)$ variables $\{w_{ij}\}$ (though this was not obviously stated). But the correctness of this approach would not be easily seen. For this, we need the following lemma.

Lemma 1. Let d be a positive integer and Q_n, Q be probability measures defined on $(R^d, B(R^d))$ such that $Q_n \xrightarrow{w} Q$. Then there is a probability space (Ω, F, P) on which we can define a sequence of random vectors $\{X_n\}$ and X such that $X_n(\omega) \to X(\omega)$, $\forall\, \omega \in \Omega$, and that X_n and X have distributions F_n and F, respectively.

In fact, this lemma is a special case of Skorohod representation theorem (1956). A further generalization can be find in Bai and Liang (1985)

Applying this lemma, we can define $\{\tilde{u}_{ij}^{(N)}, w_{ij};\ i, j = 1, 2, \cdots, p, N = 1, 2, \cdots\}$ on some probability space such that $\{u_{ij}^{(N)}\} \xrightarrow{\text{pointwise}} \{w_{ij}\}$ as $N \to \infty$, $\{u_{ij}^{(N)}\}$ and $\{\tilde{u}_{ij}^{(N)}\}$ are identically distributed and that $\{w_{ij}\}$ is a set of iid. $N(0, 1)$ random variables. Since $\{y_{ij}\}$ is independent of $\{u_{ij}^{(N)}\}$, we can also assume $\{y_{ij}\}$ is independent of $\{\tilde{u}_{ij}^{(N)}\}$. For the sake of simplicity of notation, we still use $\{u_{ij}\}$ instead of $\{\tilde{u}_{ij}^{(N)}\}$. This is to say, we assume that

$$U \to W = \begin{pmatrix} \sqrt{2}\, w_{11} \cdots w_{1p} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ w_{p1} \quad \cdots \sqrt{2}\, w_{pp} \end{pmatrix} \quad \forall\, \omega \in \Omega \tag{11}$$

(Note that, for different n, all the U's do not still have the relations determined by (8) and (10).

Lemma 2. Let $K \geqslant k$ be two positive integers. Suppose that $f_n(z) = a_K^{(n)} z^K + \cdots + a_0^{(n)} \to f(z) = a_k z^k + \cdots + a_0$, where $a_K^{(n)} \neq 0$, $a_k \neq 0$, $n = 1, 2, \cdots$. Let $z_1, \cdots, z_k$ denote the roots of f. Then we can suitably arrange the K roots of f_n as $z_1^{(n)}, \cdots, z_k^{(n)}, z_{k+1}^{(n)}, \cdots, z_K^{(n)}$ such that

$$z_i^{(n)} z_i \quad \text{for} \quad i \leqslant k, \quad |z_i^{(n)}| \to \infty \quad \text{for} \quad i > k$$

as $n \to \infty$.

Proof. If $K > k$, then $a_K^{(n)} \to 0$ and $a_k^{(n)} \to a_k \neq 0$, hence $|a_k^{(n)} / a_K^{(n)}| \to \infty$. By Weida Theorm there must be a sequence of roots, say $z_K^{(n)}$ which tends to infinity. Thus, there must be a $K-1$ degree polynomial $P_{K-1}^{(n)}(z)$, for each n, such that $f_n(z) = (1 - \frac{z}{z_K^{(n)}}) P_{K-1}^{(n)}(z)$. Noting $f_n \to f$, we get $P_{K-1}^{(n)}(z) \to f(z)$ as $n \to \infty$. By induction, we can find that there are $K-k$ roots $z_j^{(n)}$, $j = k+1, \cdots, K$, such that $|z_j^{(n)}| \to \infty$, also there is a k-degree polynomial $P_k^{(n)}(z) \to f(z)$, and all the k roots of $P_k^{(n)}(z)$ are the remaining roots of f_n.

We claim that there must be a root of $P_k^{(n)}(z)$, say $z_1^{(n)}$, such that $z_1^{(n)} \to z_1$ as $n \to \infty$. Otherwise, there must be a positive number ε_0 such that

$$\min_{1<i<k} |z_i^{(n)} - z_1| \geqslant \varepsilon_0 > 0$$

holds for infinitely many n. Let $P_k^{(n)}(z) = b_k^{(n)} \prod_{l=1}^{k} (z - z_l^{(n)})$, where $b_k^{(n)}$ is the is the coefficient of first term of $P_k^{(n)}(z)$. On one hand, $P_k^{(n)}(z_1) \to 0$ as $n \to \infty$, since $P_k^{(n)}(z_1) \to f(z_1) = 0$, as $n \to \infty$. On the other hand,

$$|P_k^{(n)}(z_1)| = |b_k^{(n)}| \prod_{l=1}^{k} |z_1 - z_l^{(n)}| \geqslant |b_k^{(n)}| \varepsilon_0^k \text{ for infinitely many } n.$$

Note $|b_k^{(n)}| \to |a_k| > 0$ and we get a contradiction. Thus our assertion is proved. By decomposition theorem of polynomial there is a $k-1$ degree polynomial $P_{k-1}^{(n)}(z)$ such that

$$P_k^{(n)}(z) = (z - z_1^{(n)}) P_{k-1}^{(n)}(z) \text{ and } P_{k-1}^{(n)}(z) \to P_{k-1}(z) \text{ as } n \to \infty,$$

where $P_{k-1}(z)$ is a $k-1$ degree polynomial such that

$$f(z) = (z - z_1) P_{k-1}(z).$$

By induction, we prove the Lemma.

Split W into blocks according to the fashion of split of C. Write the blocks of W as W_{hg}. Since $U \to W$, $\forall\ \omega \in \Omega$, by lemma 2 we know that the l_1

positive roots ϕ_i (excluding the $p-l_1$ multiplicity of root zero) satisfy

$$\phi_1 = \lambda_h + O(1), \quad i = a_{h-1}+1, \cdots, a_h, \quad h = 1,2,\cdots, \quad \nu+1 \tag{12}$$

where $\lambda_{\nu+1} = 0$, $a_{\nu+1} = l_1$. Set $\Delta = \min_{1<h<\nu} (\lambda_h - \lambda_{h+1}) > 0$. In what follows, we fix $\omega \in \Omega$. Substituting $\phi = \lambda_1 + v\eta$ into the lefthand side of (9), dividing by $v^{\frac{1}{2}}$ the first μ_1 rows and the first μ_1 columns of the determinantal equation in (9), and letting $N \to \infty$, we find the lefthand side of (9) tends to

$$\det \begin{vmatrix} C_{11} - \lambda_1 W_{11} - \eta I_{\mu_1} & & & & \\ & (\lambda_2 - \lambda_1) I_{\mu_2} & & & \\ & & \ddots & & \\ & & & (\lambda_\nu - \lambda_1) I_{\mu_\nu} & \\ & & & & -\lambda_1 I_{p-r} \end{vmatrix}$$

which is a μ_1-degree polynomial in η and whose roots are the same as that of the following equation

$$\det \| C_{11} - \lambda_1 W_{11} - \eta I_{\mu_1} \| = 0 \tag{13}$$

By (12) we know when N large enough

$$|\varphi_i - \lambda_h| < \frac{1}{3}\Delta, \quad i = a_{k-1}+1, \cdots, a_h, \quad h = 1,2,\cdots, \quad \nu+1.$$

Write $\bar{\eta}_i = (\varphi_1 - \lambda_1)/v$ and denote by $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_{\mu_1} > 0$ the roots of (13).

Then we have

$$\begin{aligned} |\bar{\eta}_i| &< \Delta/3v \quad & i &= 1,2,\cdots, \mu_1 = a_1 \\ \bar{\eta}_i &< -2\Delta/3v \quad & i &= a_1+1, \cdots, l_1 \end{aligned}$$

By Lemma 2 we know that

$$\begin{aligned} \bar{\eta}_i &\to \eta_i, \quad & i &= 1,2,\cdots, \mu_1 = a_1 \\ \bar{\eta}_i &\to -\infty, \quad & i &= a_1+1, \cdots, l_1. \end{aligned}$$

Consider the $\mu_1 \times \mu_1$ matrix $C_{11} - \lambda_1 W_{11}$. The diagonal elements of this matrix are

$$2\sqrt{\lambda_1} y_{ii} - \sqrt{2}\lambda_1 w_{ii} \sim \sqrt{2\lambda_1^2 + 4\lambda_1}\, N(0, 1),$$

and the off-diagonal elements are

$$\sqrt{\lambda_1}\,(y_{ij} + y_{ij}) - \lambda_1 u_{ij} \sim \sqrt{\lambda_1^2 + 2\lambda_1}\, N(0, 1).$$

Hence, the distribution of roots of (13) is the same as that of

$$|\sqrt{\lambda_1^2+2\lambda_1}\ W_{11}-\eta I|=0\ . \tag{14}$$

set $\zeta_i=\bar{\eta}_i/\sqrt{2\lambda_1^2+4\lambda_1}\rightarrow\eta_i/\sqrt{2\lambda_1^2+4\lambda_1}$, $i=1,2,\cdots,\ \mu_1$, as $N\rightarrow\infty$.

Following the same lines as the proof in [5] , the distribution of ζ_i, $i=1,2,\cdots,\ \mu_1$ tends to that as stated in Hsu's Theorem.

Similarly, we can prove that if write

$$\bar{\eta}_i=(\phi_i-\lambda_h)/v\ ,\ i=a_{h-1}+1\ ,\ \cdots,\ a_h,\ h=1,2,\cdots,\ v,$$

then

$$\bar{\eta}_i\rightarrow\eta_i \text{ as } N\rightarrow\infty$$

where η_i, $i=a_{h-1}+1$, $\cdots$, a_h, are the roots of the determinantal equation

$$|C_{hh}-\lambda_h W_{hh}-\eta I_{\mu_h}|=0$$

Set

$$\bar{\zeta}_i=\bar{\eta}_i/\sqrt{2\lambda_h^2+4\lambda_h}\ ,\ i=a_{h-1}+1\ ,\ \cdots,\ a_h\ .$$

Their joint distribution tends to that asstated in Hsu's Theorem.

Finally, letting $\phi=v^2\zeta$ and substituting it into (9) , dividing the last $p-r$ rows and the last $p-r$ columns of the determinantal on the lefthand side of (9) , then letting $N\rightarrow\infty$, we obtain

$$\det\begin{bmatrix}\lambda_1 I_{\mu_1} & 0 & \cdots & 0 & E_1' \\ 0 & \lambda_2 I_{\mu_2} & \cdots & 0 & E_2' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_v I_{\mu_v} & E_v' \\ E_1 & E_2 & \cdots & E_v & \bar{A}-\zeta I\end{bmatrix}=0 \tag{15}$$

where $\bar{A}$ is the lower-right $(p-r)x(p-r)$ submatrix of A. (15) is equivalent to

$$\det\left(\bar{A}-\frac{1}{\lambda_1}E_1E_v'-\cdots-\frac{1}{\lambda_v}E_vE_v'-\zeta I\right)=0 \tag{16}$$

On recalling the elements of $\bar{A}$ and E_h , (16) is in fact the following equation

$$\det\|e_{ij}-\zeta\delta_{ij}\|=0 \tag{17}$$

Where $\zeta_{ii}=1$, $\delta_{ij}=0$ (for $i\neq j$) and

$$e_{ij} = \sum_{\beta = r+1}^{n_1} y_{i\beta} y_{j\beta}.$$

Write the roots of (17) as $\zeta_{r+1}, \cdots, \zeta_{l_1}, 0, \cdots 0$. (Note the rank of $\|e_{ij}\|$ is $l_1 - r$, hence (17) has $p - l_1$ multiplicity of root zero.) Set $\overline{\zeta_i} = \phi_i / v^2$, $i = r+1, \cdots, l_1$. By Lemma 2 we know

$$\overline{\zeta_i} \to \zeta_i, \quad i = r+1, \cdots, l_1.$$

According to the proof of Hsu, we know the distribution of $\overline{\zeta_i}$, $i = r+1, \cdots, l_1$ tends to that as stated in Hsu's Theorem.

The independence among each group of limits of the corresponding groups of roots is obvious.

The proof of Hsu's Theorem is thus completed.

References

[1] BAI, Z. D. and LIANG, W. Q. (1985), Strong representation of weak convergence, Center for Multivariate Analysis, University of Pitts urgh Technical Report. No. 85-29.

[2] CHEN, X. R. (1981), Berry-Esseen bound of the error variance estimates in linear models, Sintia Scinica, 2, 129-140.

[3] FISHER, R. A. (1939), The sampling distribution of some statistics obtained from nonlinear equations, Ann. Eugen., 9, 238—249.

[4] HSU, P. L. (1939), On the distribution of roots of certain determinantal equations, Ann. Eugen., 9, 250—258.

[5] HSU, P. L. (1941), On the limiting distribution of roots of a determinantal equation, . London Math. Soc., 16, 183-194.

[6] Roy, S. N. (1939), p-statistics of some generalizations in the analysis of variance appropriate to multivariate problems, Sankhyā, 4, 381—396,

[7] KRISHNAIAH, P. R. (1982), Selection of variables in discriminant analysis, In Handbook of Statistics, Volume 2 (P. R. Krishnaiah, editor), 805—820, North-Holland Publishing Company.

[8] Skorohod, A. V. (1956), Limit theorems for stochastic processes, Theory of Prob. and its Appl., Vol. 1, No. 3, 261-290.