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A Ring with Left Identities anb Right Inverses Is a Skew Field*

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It is well known[1,2,3,5] that a semigroup in which every element has a left identity and right inverses need not be a group. In this note we prove that if the semigroup is the multiplicative semigroup of a ring, then it is a group. In fact, we have the following theorem.

Theorem Let R be a ring. Let $E=\{e\in R \mid er=r,\ \forall r\in R\}$. If E is nonempty and for each $r\in R$, $r\neq 0$, there is at least one element $r'\in R$ such that $rr'\in E$, then R is a skew field.

Proof It is easily seen that $0\notin E$ except $R=\{0\}$. If there are two elements $e_1, e_2\in E$, then $e_1r=r$; $e_2r=r$ for all $r\in R$. Hence $e_1r-e_2r=0$, $(e_1-e_2)r=0$, for all $r\in R$. Suppose $e_1-e_2\neq 0$, then there is at least one element $k\in R$ such that $(e_1-e_2)k\in E$, a contradiction to the fact that $(e_1-e_2)r=0$ for all $r\in R$, therefore $e_1-e_2=0, e_1=e_2$.

Now we prove that a right inverse in R is also a left inverse. Let $r\in R$, $r\neq 0$. there is at least one element $r'\in R$ such that $rr'=e$. Hence $(rr')r=er$, $r(r'r)=r$, $r'r\neq 0$. there is at least one element $h\in R$ such that $(r'r)h=e$. From $r(r'r)=r$ we know $r'[r(r'r)]=r'r$, $(r'r)(r'r)=r'r$. Multiplying the last equality from the right by h, we have $(r'r)e=e$, $[(r'r)-e]e=0$. If $(r'r)-e\neq 0$, there is at least one element $t\in R$ such that $[(r'r)-e]t=e$. Multiplying $[(r'r)-e]e=0$ from the right by t, we have $e=0$, impossible, Hence $r'r-e=0$, $r'r=e$.

It is well known ([4, p4]) that a semigroup in which every element has a left indentity and left inverses is a group. Therefore $R\setminus\{0\}$ is a group, R is a skew field.

References

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