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Characterization of Polynomial Systems Satisfying Generalized Convolution Differential-Difference Equations*

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1. Introduction. The present paper has been motivated by the desire to find all polynomial solutions of the convolution type differential -difference equation

$$(1.1)\qquad D_x g_n(x) = \sum_{i=1}^{n-1} g_i(x) g_{n-i}(x), \quad n \geq 2,$$

where $g_1(x)$ is assumed to be a constant. This problem arose in work by one of the authors (Kerr) with a differential equation arising in a coal research project [11]. We shall first solve this equation and then consider some more general relations suggested by (1.1). Relation (1.1) may be rewritten in the suggestive form

$$(1.2)\qquad D_x g_n(x) = \sum_{\substack{i+j=n \\ 1\le i,\, j\le n}} g_i(x) g_j(x),$$

where the summation is over all integer solutions i, j satisfying $i+j=n$ and such that $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. The reason this is suggestive is that in Section 3 we analyze the equation

$$(1.3)\qquad D_x U_{n+1}(x) = \sum_{\substack{i+j=n \\ 0\le i,\, j\le n}} U_i(x) U_j(x)$$

and in Section 4 the equation

$$(1.4)\qquad D_x X_{n+1}(x) = \sum_{\substack{i+j+k=n \\ 0\le i,\, j,\, k\le n}} X_i(x) X_j(x) X_k(x),$$

giving solutions under some simple hypotheses.

We also extend our consideration of (1.3) and (1.4) to equations involving higher derivatives. If we impose equation (1.3) together with the equation

$$(1.5)\qquad D_x^k U_{n+k}(x) = k! \sum_{\substack{j_1+j_2+\cdots+j_{k+1}=n \\ 0\le j_i\le n}} U_{j_1}(x) U_{j_2}(x) \cdots U_{j_{k+1}}(x)$$

we obtain a general solution that includes the case where $U_n(x)$ is nothing but the coefficient of t^n in the expansion of the generating series $(c-tx)^{-1}$ for a suitable

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constant c. We also determine a linear recurrence relation for the general $U_n(x)$ polynomials.

We can extend our consideration of (1.4) by imposing the higher derivative equation

(1.6) $$D_x^k X_{n+k}(x) = (2k-1)!! \sum_{\substack{j_1+j_2+\cdots+j_{2k+1}=n \\ 0\le j_i\le n}} X_{j_1}(x)X_{j_2}(x)\cdots X_{j_{2k+1}}(x)$$

where the 'double factorial' symbol means $1\cdot3\cdot5\cdots(2k-1)$. We find that the polynomials $X_n(x)$ are now the coefficients of t^n in the series expansion of the generating function $(c-2tx)^{-1/2}$. Among the special solutions we then note that the choice $c=1+t^2$ gives $X_n(x)=P_n(x)$, the well-known Legendre polynomials.

The fact that the Legendre polynomials satisfy (1.4 was found by Catalan [2], [3]. Djokovic [4] found (1.6) true for Legendre polynomials but proved it only for the cases $k=1,2,3$. Boas [1] then gave a simple proof of (1.6) for Legendre polynomials. Scott [17] found a related result for Tchebycheff polynomials of the second kind. Popov [12] also proved that (1.6) holds for Legendre polynomials. Gould [7] found still another related identity for a species of generalized Humbert polynomials. In [7] the above history is recounted. We use the method [9] of the algebra of formal power series. In Section 5 below we show how to find nonlinear recurrences for Bernoulli numbers.

2. Solution of (1.1) We introduce the generating function $G(x,t)$ defined by

(2.1) $$G(x,t)=\sum_{n=1}^{\infty} g_n(x)t^n.$$

Then $$D_xG(x,t)=\sum_{n=1}^{\infty} t^n D_x g_n(x) = tD_x g_1(x)+\sum_{n=2}^{\infty} t^n D_x g_n(x)$$

$$= tg_1'(x)+\sum_{n=2}^{\infty} t^n \sum_{k=1}^{n-1} g_k(x)g_{n-k}(x) = tg_1'(x)+\left\{\sum_{n=1}^{\infty} t^n g_n(x)\right\}^2 = tg_1'(x)+G^2(x,t),$$

so that the use of our formal power series yields the differential equation

(2.2) $$D_xG(x,t)=tg_1'(x)+G^2(x,t).$$

This equation is a special form of the general Riccati equation

(2.3) $$y'=\alpha(x)y^2+\beta(x)y+\gamma(x).$$

What we need to know about (2.3) is to be found in Kamke [10], Murphy [13] and Watson [18].

We consider first the case where $g_1(x)$ is constant so that $g_1'(x)=0$. Then we can solve (2.2) at once and get

(2.4) $$G(x,t)=\frac{1}{c-x}, \text{ where } c \text{ is a constant}.$$

A clever way to make use of the constant c is to proceed as follows. Let

(2.5) $$C(t)=G(0,t)=\sum_{n=1}^{\infty} c_n t^n,$$

and then by (2.4) we have $G(0,t)=1/c$ and $C_1=g_1(x)$.
From this it follows that the solution of (2.2) when $g_1(x)=c_1$, a constant, is given by

(2.6) $$G(x,t)=\frac{C(t)}{1-xC(t)}$$

in terms of a formal power series (2.5) where c_n is an arbitrary sequence of real numbers. If we choose a function $C(t)$ we can then expand $G(x,t)$ in a formal power series and find $g_n(x)$ as the coefficient of t^n.

But let us find $g_n(x)$ directly in terms of the c_i's as follows, We have

$$G(x,t)=C(t)\{1-xC(t)\}^{-1}=C(t)\sum_{k=0}^{\infty}x^k\{C(t)\}^k=\sum_{k=0}^{\infty}x^k\{C(t)\}^{k+1}=\sum_{k=0}^{\infty}x^k\{\sum_{n=1}^{\infty}c_n t^n\}^{k+1}$$

$$=\sum_{k=0}^{\infty}x^k\sum_{n=k+1}^{\infty}t^n\sum_{\substack{j_1+j_2+\cdots+j_{k+1}=n\\ 1\le j_i\le n}}c_{j_1}c_{j_2}\cdots c_{j_{k+1}}=\sum_{n=1}^{\infty}t^n\sum_{k=0}^{n-1}x^k\sum_{\substack{j_1+\cdots+j_{k+1}=n\\ 1\le j_i\le n}}c_{j_1}c_{j_2}\cdots c_{j_{k+1}}$$

whence, recalling (2.1) we have that the g's are polynomials and in fact

(2.7) $$g_n(x)=\sum_{k=0}^{n-1}b_{n,k}x^k$$

where the coefficients $b_{n,k}$ are determined by

(2.8) $$b_{n,k}=\sum_{\substack{j_1+\cdots+j_{k+1}=n\\ 1\le j_i\le n}}c_{j_1}c_{j_2}\cdots c_{j_{k+1}}$$

Below we give a short table of the first few values of the b's.

$n \backslash k$	0	1	2	3	4
1	c_1	0	0	0	0
2	c_2	c_1^2	0	0	0
3	c_3	$2c_1c_2$	c_1^3	0	0
4	c_4	$2c_1c_3+c_2^2$	$3c_1^2c_2$	c_1^4	0
5	c_5	$2c_1c_4+2c_2c_3$	$3c_1^2c_3+3c_1c_2^2$	$4c_1^3c_2$	c_1^5

Let us examine a few simple examples of the applications of the formulas.

Example I. Let $c_i=a$ for all $i\geqslant 1$. Then

$$b_{n,k}=\sum_{\substack{j_1+\cdots+j_{k+1}=n\\ 1\le j_i\le n}}a^{k+1}=a^{k+1}\sum_{\substack{j_1+\cdots+j_{k+1}=n\\ 1\le j_i\le n}}1=\binom{n-1}{k}a^{k+1}$$

by a known result (also given by Catalan) noted in [6.p.242]. Hence we find

(2.9) $$g_n(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}a^{k+1}x^k=a(1+ax)^{n-1}.$$

This may be checked in another way. Using (2.5) we get $C(t)=at(1-t)^{-1}$ so that

$$G(x,t)=(c-x)^{-1}=at(1-(1+ax)t)^{-1}=a\sum_{n=1}^{\infty}(1+ax)^{n-1}t^n,$$

so that again (and more easily) we find $g_n(x)=a(1+at)^{n-1}$.

Example 2. Let $c_i=i$ for all $i\geqslant 1$. The reader may readily check the details that here we shall have

$$g_n(x)=\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k \quad . \tag{2.10}$$

The first five of these g's are as follows:

$$g_1=1,\quad g_2=2+x,\quad g_3=3+4x+x^2,$$
$$g_4=4+10x+6x^2+x^3\ ,\ g_5=5+20x+21x^2+8x^3+x^4\ .$$

If we verify (2.10) by using the array of c's we make use of the easy to prove known identity

$$\sum_{\substack{j_1+j_2+\cdots+j_k=n\\ 1\leqslant j_i\leq n}} j_1j_2\cdots j_k=\binom{n+k-1}{2k-1}\ . \quad (1\leqslant j_i\leqslant n) \tag{2.11}$$

It is also not difficult to write (2.10) in closed form because in fact

$$\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k=\frac{a^n-b^n}{a-b},\ \text{where } ab=1,\ a+b=x+2. \tag{2.12}$$

In (2.12) the a and b numbers are the distint roots of a quadratic equation. Relation (2.12) is akin in fact to the expansion for Fibonacci numbers

$$F_n=\sum_{0\leq k\leq (n-1)/2}\binom{n-1-k}{k}=\frac{a^n-b^n}{a-b} \tag{2.13}$$

where now a and b are roots of the quadratic equation $x^2-x-1=0$. These roots are, of course, $(1\pm\sqrt{5})/2$ and the expression $(a^n-b^n)/(a-b)$ is the so-called Binet formula for Fibonacci numbers.

We return to the general situation and prove next that the g_n's which satisfy relation (1.1) with $g_1(x)=c_1=$constant also satisfy the linear recurrence relation

$$g_n(x)=c_n+x\sum_{j=1}^{n-1}c_{n-j}g_j(x),\quad n\geqslant 2. \tag{2.14}$$

Proof. From (2.6)

$$G(x,t)=C(t)+xC(t)G(x,t).$$

Substituting from (2.1) and (2.5) we get

$$\sum_{n=1}^{\infty}g_n(x)t^n=\sum_{n=1}^{\infty}c_nt^n+x\left(\sum_{n=1}^{\infty}c_nt^n\right)\left(\sum_{k=1}^{\infty}g_k(x)t^k\right)=\sum_{n=1}^{\infty}c_nt^n+x\sum_{n=2}^{\infty}t^n\sum_{j=1}^{n-1}c_{n-j}g_j(x),$$

whence by coefficient comparison we have evidently proved (2.14).

Relation (2.14) allows an independent way of generating as many g's as we need from those we already have found in any given case.

Remark. It is clear by induction, with $g_1(x)=$constant, that (2.2) yields

$$D_x^kG(x,t)=k!\,G^{k+1}(x,t). \tag{2.15}$$

But by (2.5) and (2.1)

$$G^{k+1}(x,t)=\{\sum_{n=1}^{\infty}g_n(x)t^n\}^{k+1}=\sum_{n=k+1}^{\infty}t^n\sum_{\substack{j_1+\cdots+j_{k+1}=n\\1\le j_i\le n}}g_{j_1}(x)\cdots g_{j_{k+1}}(x)$$

whence we have proved the interesting extension of (1.1)

$$D_x^k g_n(x)=k!\sum_{\substack{j_1+\cdots+j_{k+1}=n\\1\le j_i\le n}}g_{j_1}(x)\cdots g_{j_{k+1}}(x),\quad k\geqslant 0. \tag{2.16}$$

Of course, we have derived all under the assumption $g_1(x)=\text{constant}$. The situation for $g_1(x)$ nonconstant is, of course, subject to what follows from the Riccati equation (2.2).

We conclude this section of our paper with a derivation of another recurrence relation which is as follows:

$$\sum_{k=1}^{n-1}kk_{n-k}g_k(x)=\sum_{k=1}^{n-1}kc_kg'_{n-k}(x), \tag{2.17}$$

where

$$k_n=\sum_{j=1}^{n-1}c_jc_{n-j}\ ,\quad \text{with}\quad k_1=0. \tag{2.18}$$

Proof. From (2.6) it is easy to obtain the partial differential equation

$$C^2(t)D_tG(x,t)=C'(t)D_xG(x,t)\ . \tag{2.19}$$

In this make the indicated substitutions using (2.1) and (2.5). The result is that

$$\sum_{n=2}^{\infty}t^nk_n\cdot\sum_{n=1}^{\infty}nt^{n-1}g_n(x)=\sum_{n=1}^{\infty}nt^{n-1}c_n\cdot\sum_{n=1}^{\infty}g_n'(x)t^n\ ,$$

and simplifying and equating coefficients of powers of t we find (2.17).

This is the simplest linear recurrence we have found between the g's and the derivatives of the g's.

3. Equation (1.3) **and its extension to** (1.5). Proceeding as we did with (1.1) we introduce the generating function

$$H(x,t)=\sum_{n=0}^{\infty}t^nU_n(x)\ . \tag{3.1}$$

Then

$$H^2=\sum_{n=0}^{\infty}t^n\sum_{\substack{i+j=n\\0\le i,j\le n}}U_i(x)U_j(x)=\sum_{n=0}^{\infty}t^nD_xU_{n+1}(x)=t^{-1}D_x\sum_{n=1}^{\infty}t^nU_n(x)=t^{-1}D_x\{H(x,t)-U_0(x)\}$$

whence our Riccati type differential equation now reads

$$D_xH(x,t)=tH^2(x,t)+D_xU_0(x)\ . \tag{3.2}$$

In the case that $U_0(x)$ is constant, this is readily solved to give

$$H(x,t)=\frac{1}{c-tx},\quad \text{for a suitable constant } c. \tag{3.3}$$

Proceeding as we did in Section 2, we let

(3.4) $$H(0,t)=C(t)=\sum_{n=0}^{\infty}c_n t^n=\frac{1}{c},$$ and we can then write (3.3) in the form

(3.5) $$H(x,t)=\frac{C(t)}{1-txC(t)}$$

This then yields

$$H(x,t)=C(t)\{1-txC(t)\}^{-1}=\sum_{k=0}^{\infty}t^k x^k\{C(t)\}^{k+1}=\sum_{k=0}^{\infty}t^k x^k\ \{\sum_{n=0}^{\infty}c_n t^n\}^{k+1}$$

$$=\sum_{k=0}^{\infty}t^k x^k\sum_{n=0}^{\infty}t^n\sum_{\substack{j_1+\cdots+j_{k+1}=n\\0\le j_i\le n}}c_{j_1}\cdots c_{j_{k+1}}=\sum_{k=0}^{\infty}t^k x^k\sum_{n=k}^{\infty}t^{n-k}\sum_{j_1+\cdots+j_{k+1}=n-k}c_{j_1}\cdots c_{j_{k+1}}$$

$$=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}x^k\sum_{j_1+\cdots+j_{k+1}=n-k}c_{j_1}\cdots c_{j_{k+1}}$$

whence we have proved that

(3.6) $$U_n(x)=\sum_{k=0}^{n}x^k\sum_{\substack{j_1+\cdots+j_{k+1}=n-k\\0\le j_i\le n-k}}c_{j_1}c_{j_2}\cdots c_{j_{k+1}}$$

A very simple special case is when $c_i=1$ for all $i\geqslant 0$, Then it is easy to see that $U_n(x)=(1+x)^n$.

The linear recurrence relation analogous to (2.14) is now

(3.7) $$U_{n+1}(x)=c_{n+1}+x\sum_{k=0}^{n}c_{n-k}U_k(x)\quad \text{for } n\geqslant 0$$

which is readily proved by writing (3.4) in the form

$$H(x,t)=C(t)+txC(t)H(x,t)$$

and making the indicated substitutions from (3.1) and (3.4), then expanding and comparing coefficients.

In what we have done so far we considered the case $U_0(x)$ constant. Discard this assumption. Let us assume only that we have (1.3) together with its implication (3.2). If we impose the additional hypothesis (1.5) we will prove the remarkable result that then $D_x^k U_{k-1}(x)=0$ for every integer $k\geqslant 1$.This will make $U_n(x)$ automatically a polynomial of degree n in x and force $D_x^n H(x,t)=t^n n!\,H^{n+1}(x,t)$. As a consequence the functions that satisfy both (1.3) and (1.5) are rather special .

We first prove that

(3.8) $$D_x^k H(x,t)=t^k k!\,H^{k+1}(x,t)+\sum_{n=0}^{k-1}t^n D_x^k U_n(x),\quad k\geqslant 0.$$

This is vacuously true for $k=0$. By assuming (3.2) we find

$$\sum_{n=0}^{\infty}t^n D_x^k U_{n+k}(x)=k!\ \sum_{n=0}^{\infty}t^n\sum_{\substack{j_1+\cdots+j_{k+1}=n\\0\le j_i\le n}}U_{j_1}(x)\cdots U_{j_{k+1}}(x)=k!\ \{\sum_{n=0}^{\infty}t^n U_n(x)\}^{k+1}=k!\,H^{k+1}(x,t).$$

Replace n by $n-k$ in the left-hand sum and we have then

$$t^k k!\,H^{k+1}(x,t)=\sum_{n=k}^{\infty}t^n D_x^k U_n(x)=\sum_{n=0}^{\infty}t^n D_x^k U_n(x)-\sum_{n=0}^{k-1}t^n D_x^k U_n(x)$$

which says what we gave as (3.8).

We may now state our intended result in the following form

Theorem. The functions $U_n(x)$, $n=0,1,\cdots$ arising in relation (3.8) are polynomials in x of degree $\leq n$.

Proof. We will show first that

$$\sum_{n=0}^{k-1}t^n D_x^k U_n(x)=0 \quad \text{for all } k\geqslant 1. \tag{3.9}$$

Lemma. If (3.8) holds true for any $k\geqslant 1$, where x and t are not constants, and $H(x,t)$ is not identically constant for all t, then

$$D_x^k H(x,t)=t^k k!\,H^{k+1}(x,t) \quad \text{for all } k\geqslant 1. \tag{3.10}$$

Proof. We prove this by induction on k. To begin with, when $k=1$ in (3.8) we obtain (3.2) and when $k=2$ we get

$$D_x^2 H(x,t)=2t^2H^3(x,t)+D_x^2U_0(x)+tD_x^2U_1(x). \tag{3.11}$$

Then, differentiating (3.2) we have

$$D_x^2 H(x,t)=2tH(x,t)D_xH(x,t)+D_x^2U_0(x)$$

and substituting with (3.2) we get

$$D_x^2 H(x,t)=2t^3H^3(x,t)+2tH(x,t)D_xU_0(x)+D_x^2U_0(x). \tag{3.12}$$

Now subtract (3.11) from (3.12) and we get the relation

$$2tH(x,t)D_xU_0(x)=tD_x^2U_1(x)$$

holding for all t so

$$2H(x,t)D_xU_0(x)=D_x^2U_1(x). \tag{3.13}$$

Since t is not constant and $H(x,t)$ is not identically constant for all t we may assume there exist t_1 and t_2 such that $H(x,t_1)\neq H(x,t_2)$. Then by (3.13) we know that $2H(x,t_1)D_xU_0(x)=D_x^2U_1(x)$ and $2H(x,t_2)D_xU_0(x)=D_x^2U_1(x)$ from which we get that $2D_xU_0(x)\{H(x,t_1)-H(x,t_2)\}=0$, and this, of course, implies that $D_xU_0(x)=0$. Then (3.13) tells us that $D_x^2U_1(x)=0$,

The result that $D_xU_0(x)=0$. means next that (3.2) reduces to

$$D_xH(x,t)=tH^2(x,t) \tag{3.14}$$

which establishes (3.10) when $k=1$.

Now, suppose that (3.10) holds for $k=r$ (some natural number). Then

$$D_x^rH(x,t)=t^r r!\,H^{r+1}(x,t)$$

Differentiating this yields $D_x^{r+1}H(x,t)=t^r(r+1)!\,H^r(x,t)D_xH(x,t)$, so that by (3.14) we have $D_x^{r+1}H(x,t)=t^{r+1}(r+1)!\,H^{r+2}(x,t)$. We have therefore shown that (3.10) holds for $k=r+1$. By the principle of mathematical induction we have shown then that (3.10) is valid for all $k\geqslant 1$. Q. E. D.

Combining this with (3.8) we have therefore established (3.9). However, note that (3.9) is a polynomial equation in t holding for an infinite number of t's, but this contradicts the Fundamental Theorem of Algebra since (3.9) could hold for no more than $k-1$ values of t. Hence the coefficients must vanish identically so that $D_x^k U_{-1}(x)=0$ for every $k \geqslant 1$. Thus our theorem is proved.

The implication of this result is that if we impose both (1.3) and (1.5) then we obtain simple polynomials for $U_n(x)$. These are given by (3.6) in terms of the constants c_i or we can use the previously obtained generating function (3.3) or (3.5).

The first few U's are as follows:

$$U_0(x)=c_0, \quad U_1(x)=c_1+c_0^2x, \quad U_2(x)=c_2+2c_0c_1x+c_0^3x^2,$$
$$U_3(x)=c_3+(2c_0c_2+c_1^2)x+3c_0^2c_1x^2+c_0^4x^3,$$
$$U_4(x)=c_4+(2c_0c_3+2c_1c_2)x+(3c_0^2c_2+3c_0c_1^2)x^2+4c_0^3c_1x^3+c_0^5x^4.$$

They, of course, are formed just as the g_n's were in relation (2.8). Denote by 1 each subscript in the tables of $b_{n,k}$ following (2.8) and we obtain the above U's.

4. Equation (1.4) **and its extension to** (1.6). First of all, if we set up the generating function

$$K(x,t)=\sum_{n=0}^{\infty} t^n X_n(x) \tag{4.1}$$

and impose (1.4), then there is no difficulty in showing that the generating function must satisfy the equation

$$D_xK(x,t)=tK^3(x,t)+D_xX_0(x), \tag{4.2}$$

which is an Abel type differential equation. The general Abel equation [10], [13] has the form

$$y'=\alpha(x)y^3+\beta(x)y^2+\gamma(x)y+\delta(x). \tag{4.3}$$

In case we suppose $X_0(x)$ to be constant, then equation (4.2) has the evident solution

$$K(x,t)=\frac{1}{\sqrt{c-2tx}}, \text{ for a suitable constant } c. \tag{4.4}$$

If we choose

$$c=1+t^2 \tag{4.5}$$

then we have

$$K(x,t)=(1-2xt+t^2)^{-1/2}=\sum_{n=0}^{\infty} t^nP_n(x), \tag{4.6}$$

so that $X_n(x)=P_n(x)$, the standard Legendre polynomials.

Note however that for any constant c we have

$$D_x(c-2tx)^{-1/2}=t\{(c-2tx)^{-1/2}\}^3$$

so that Legendre polynomials are indeed a very special case.

Motivated by knowledge of (1.6) being true for Legendre polynomials, let us next carry through a proof of a theorem parallel to what we did in the previous section, by imposing both (1.4) and (1.6).

In the same manner that we found (3.8) it is a routine calculation to derive

(4.7) $$D_x^k K(x,t) = (2k-1)!!\,t^k K^{2k+1}(x,t) + \sum_{n=0}^{k-1} t^n D_x^k X_n(x),$$

out of (1.6) and (4.1) holding for all $k \geqslant 1$, so it includes (1.4) in fact. Relation (4.7) is analogous to (3.8). We are now in the position to prove the

Theorem. The functions $X_n(x)$ arising in relation (4.7) are polynomials in x of degree at most n.

Proof. Just as we proved (3.9), we begin by showing that

(4.8) $$\sum_{n=0}^{k-1} t^n D_x^k X_n(x) = 0 \quad \text{for all } k \geqslant 1.$$

To get this we need the

Lemma. If (4.8) holds for non-constant x, t and $K(x,t)$ is not identically constant for all t, then

(4.9) $$D_x^k K(x,t) = (2k-1)!!\,t^k K^{2k+1}(x,t) \quad \text{for all } k \geqslant 1.$$

Proof. We use induction on k. To begin with, when $k=1$ (4.7) has the form

(4.10) $$D_x K = tK^3 + D_x X_0(x),$$

and when $k=2$ we have

(4.11) $$D_x^2 K = 3t^2K^5 + D_x^2 X_0(x) + tD_x^2 X_1(x).$$

Differentiating (4.10) once more yields $D_x^2 K = 3tK^2 D_x K + D_x^2 X_0(x)$, whereby upon substitution of (4.10) we get

(4.12) $$D_x^2 K = 3t^2K^5 + 3tK^2 D_x X_0(x) + D_x^2 X_0(x).$$

Now by combining (4.11) and (4.12) we get the relation

$$3tK^2 D_x X_0(x) = tD_x^2 X_1(x),$$

which holds for all non-constant t. Hence

(4.13) $$3K^2 D_x X_0(x) = D_x^0 X_1(x).$$

Again we may suppose that there exist different t_1, t_2 such that $K(x,t_1) \neq K(x,t_2)$, and in fact $K^2(x,t_1) \neq K^2(x,t_2)$. Therefore we have both $3K^2(x,t_1)D_x X_0(x) = D_x^2 X_1(x)$ and $3K^2(x,t_2)D_x X_0(x) = D_x^2 X_1(x)$, which combine to give

$$3D_x X_0(x) \cdot \{K^2(x,t_1) - K^2(x,t_2)\} = 0$$

which entails then that $D_x X_0(x) = 0$. Putting this back into (4.13) then gives us $D_x^2 X_1(x) = 0$. As a matter of fact we have shown that (4.10) becomes just $D_x K = tK^3$ so that (4.9) is valid for $k=1$.

Suppose then that (4.9) is true for $k=r$ (some natural number). Then $D_x^r K = (2r-1)!!\,t^r K^{2r+1}$ and by differentiation of this we get

$$D_x^{r+1} K = (2r-1)!!\,t^r(2r+1)K^{2r} D_x K = (2r+1)!!\,t^r K^{2r} tK^3 = (2r+1)!!\,t^{r+1}K^{2r+3},$$

so the induction goes through and we have proved that (4.9) is true for all $k \geqslant 1$.

The remainder of our proof of our theorem is parallel to what we did in Section 3; for relation (4.8) then implies that $D_x^k X_n(x)=0$ for every $k\geqslant 1$.

5. Recurrences for Bernoulli numbers. In Section 2 we concentrated on the case where $g_1'(x)=0$. We now consider the case when $g_1'(x)=g$, a non-zero constant. In this case the Riccati equation [13, p.229]

$$y'=a+by^2, \quad a\neq 0 \tag{5.1}$$

has two solutions in terms of an arbitrary constant c:

$$ry=a\tan(C+rx), \quad r=\sqrt{ab}, \quad ab>0 \tag{5.2}$$

or

$$sy=a\tanh(C+sx), \quad s=\sqrt{-ab}, \quad ab<0. \tag{5.3}$$

We will illustrate by using (5.2)) to solve

$$D_x G(x,t)=tg+G^2(x,t), \quad tg>0. \tag{5.4}$$

We find then that

$$G(x,t)=t^{1/2}g^{1/2}\tan(C+xt^{1/2}g^{1/2}). \tag{5.5}$$

Consider the case when $C=0$, We need the well-known expansion

$$\tan z=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n)!}2^{2n}(2^{2n}-1)B_{2n} \tag{5.6}$$

where the B's are the Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n . \tag{5.7}$$

An account of formulas and history for the Bernoulli numbers may be found in [8], [14], and [16].

The result is that when $C=0$

$$G(x,t)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{(2n)!}g^n x^{2n-1}2^{2n}(2^{2n}-1)B_{2n} \tag{5.8}$$

as the formal power series for G. Thus we find that

$$g_n(x)=(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}g^n B_{2n}x^{2n-1}, \quad n\geqslant 1.\ g\neq 0. \tag{5.9}$$

The first few Bernoulli numbers are: $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, $B_8=-1/30$, $B_{10}=5/66$, $B_{12}=-691/2730$, $B_{14}=7/6$, $B_3=B_5=\cdots=0$. Thus the first few g's are:

$$g_1(x)=gx, \quad g_2(x)=\frac{1}{3}g^2x^3, \quad g_3(x)=\frac{2}{15}g^3x^5, \tag{5.10}$$

$$g_4(x)=\frac{17}{315}g^4x^7, \quad g_5(x)=\frac{62}{2835}g^5x^9, \quad g_6(x)=\frac{1382}{155925}g^6x^{11}.$$

Since the g's satisfy the original equation (1.1), then we have evidently shown that the Bernoulli numbers satisty the nonlinear recurrence relation

$$\sum_{k=1}^{n-1}\binom{2n}{2k}(2^{2k}-1)(2^{2n-2k}-1)B_{2k}B_{2n-2k}=-(2n-1)(2^{2n}-1)B_{2n}, \text{ for } n\geqslant 2. \tag{5.11}$$

which may be contrasted with the more well-known recurrence

$$\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}=-(2n+1)B_{2n},\quad n\geqslant 2. \tag{5.12}$$

Multiplying out the powers of 2 and combining (5.11) with (5.12) we get next

$$\sum_{k=1}^{n-1}\binom{2n}{2k}2^{2k}B_{2k}B_{2n-2k}=-(2n+2^{2n})B_{2n},\quad n\geqslant 2. \tag{5.13}$$

Actually the Bernoulli numbers satisfy linear recurrences, for example the well-known relation

$$\sum_{k=0}^{n}\binom{n}{k}B_k=B_n,\quad n\geqslant 0, \text{but}\quad n\neq 1. \tag{5.14}$$

Relation (5.11) is given by Saalschütz [16, p.17, No.XII] and Nielsen [14, p.67, No.15]. Further, relation (5.12) is given by Saalschütz (p.16, No.XI) and Nielsen (p.66, No.13). Finally, (5.13) is given by Nielsen (p.67, No.21). Saalschütz traces these formulas back to Euler in fact.

We think it interesting that our technique leads to these nonlinear recurrences. Many other nonlinear recurrences may be found from the expansions we have studied here.

We remark that pursuit of the hyperbolic tangent solution (5.3) yields the same Bernoulli number recurrences.

Because of the well-known formula

$$\zeta(2n)=(-1)^{n-1}\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_{2n},\quad n\geqslant 1, \tag{5.15}$$

connecting Bernoulli numbers and the Riemann Zeta function evaluated at even integers, then (5.11), (5.12), (5.13) may be restated as recurrences for this function:

$$\sum_{k=1}^{n-1}(2^{2k}-1)(2^{2n-2k}-1)\zeta(2k)\zeta(2n-2k)=(n-\frac{1}{2})(2^{2n}-1)\zeta(2n),\ n\geqslant 2. \tag{5.16}$$

$$\sum_{k=1}^{n-1}\zeta(2k)\zeta(2n-2k)=(n+\frac{1}{2})\zeta(2n),\ n\geqslant 2, \tag{5.17}$$

and

$$\sum_{k=1}^{n-1}2^{2k}\zeta(2k)\zeta(2n-2k)=(n+2^{2n-1})\zeta(2n),\ n\geqslant 2. \tag{5.18}$$

Certainly (5.17) is the simplest and most well-known.

6. A further generalization. By the methods we have outlined we can study polynomials that satisfy

$$D_x Y_{n+a+bk}(x)=f(k)\sum_{\substack{j_1+\cdots+j_{c+dk}=0\\ 0\leq j_i\leq n}}Y_{j_1}(x)Y_{j_2}(x)\cdots Y_{j_{c+dk}}(x) \tag{6.1}$$

by defining the generating function

$$W(x,t)=\sum_{n=0}^{\infty}t^nY_n(x) \tag{6.2}$$

which yields the differential equation

$$D_x^k W(x,t) = f(k) t^{a+bk} W^{c+dk}(x,t) + \sum_{n=0}^{a+bk-1} t^n D_x^k Y_n(x), \tag{6.3}$$

so that we have to deal in general with the nonlinear differential equation of form

$$y^{(k)}(x) = A(k) y^{c+dk}(x) + B(x). \tag{6.4}$$

Whenevar we are lucky enough to be able to solve this exactly we can generate interesting solutions to (6.1).

To illustrate the method we mention first $k=1$, $a=2$, $b=0$, $c=2$, $d=0$ and $f(k)=1$. Then we have to solve

$$y'(x) = t^2 y^2(x) + Y_0'(x) + tY_1'(x). \tag{6.5}$$

This equation is like (2.2). If we let $Y_0'(x)=0$ and $Y_1'(x)=$ constan $t \neq 0$ we could solve the equation as we did (5.1). Again we are solving Riccati equations. The reader can work out the details using the information here and in Kamke [10] or Murphy [13].

We will next look at the case where $k=2$, $c+dk>1$. In general this is a difficult case because many of the examples lead to solutions involving the Weierstrass elliptic function $\wp(z)$, which occurs often among solutions to nonlinear second order differential equations. Referring to Kamke [10, pp.542-544] we do not find many equations with strikingly simple solutions. For example, the equation

$$y'' = y^2 \tag{6.6}$$

may be solved in the form

$$y = \wp\left(\frac{x}{\sqrt{6}} - c_2;\ 0, c_1\right), \quad c_1 \text{ and } c_2 \text{ arbitrary constants,} \tag{6.7}$$

where $\wp(z; g_2, g_3)$ denotes the well-known Weierstrass elliptic function with invariants g_2 and g_3.

The simple equation

$$y'' = 6y^2 \tag{6.8}$$

has the solution

$$y = \wp(x + c_2;\ 0, c_1). \tag{6.9}$$

The equation

$$y'' = 6y^2 + k \tag{6.10}$$

actually characterizes the so-called Painlevé transcendental function.

There is one equation

$$y'' = Ay^3 \tag{6.11}$$

which has among its solutions

$$y = \sqrt{\frac{2}{A}} \cdot \frac{1}{x+C}, \quad C = \text{arbitrary constant}, \tag{6.12}$$

and we will examine some applications of this to obtain interesting polynomial as well as non-polynomial solutions to the appropriate specialization of (6.3).

Consider the case $k=2$, $a=0$, $b=0$, $c=3$, $d=0$, and $f(k)=1$. Then we have

$$D_x^2 Y_n(x) = \sum_{\substack{i+j+k=n \\ 0\leq i,j,k\leq n}} Y_i(x) Y_j(x) Y_k(x), \tag{6.13}$$

$$W(x,t) = \sum_{n=0}^{\infty} t^n Y_n(x), \tag{6.14}$$

$$D_x^2 W(x,t) = W^3(x,t), \tag{6.15}$$

so that we use (6.12) with $A=1$. Among the solutions of (6.15) then, we have

$$W(x,t) = \frac{\sqrt{2}}{x+C} \tag{6.16}$$

As we did in Section 2, we set $W(0,t)=C(t)=\sqrt{2}/C$ and $C(t)=\sum_{n=0}^{\infty} c_n t^n$. We can then write (6.16) in the alternative form

$$W(x,t) = \frac{\sqrt{2}\,C(t)}{\sqrt{2}+xC(t)}, \tag{6.17}$$

so that for suitable $C(t)$ or sequence c_n we may write out interesting functions satisfying (6.13).

Example 1. With $C=-1/t$ we find $Y_0=0$, $Y_n(x)=-\sqrt{2}\,x^{n-1}$, $n\geqslant 1$.

Example 2. With $C(t)=\sqrt{2}(1-t)^{-1}$ we find $W(x,t)=\sqrt{2}(x+1-t)^{-1}$ so that $Y_n(x)=\sqrt{2}(1+x)^{-n-1}$, $n\geqslant 0$.

Example 3. With $C=1-t^2$ we find $Y_{2n+1}(x)=0$, $Y_{2n}(x)=\sqrt{2}(1+x)^{-n-1}$, for $n\geqslant 0$.

Remark. A study of the identities resulting from Equations (6.6) and (6.8) which involve the Weierstrass function will be left to a separate paper.

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