Journol of Mathematical Research and Exposition
Vol.9, No.2, May, 1989

Some Bivariate Inverse Relations With Applications to Interpolation Formulae*

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Abstract

This paper investigates the bivariate analogs of Gould -Hsu's inversions and obtains an inverse relation including five sets of free parameters which can be seen as a partial solution for Hsu's open problem. Some instances and applications to interpolation theory are remarked.

I. Introduction

In an earlier paper of Gould and Hsu [1], a remarkable result has been found which may be stated as follows: Let $\{a_j\}$ and $\{b_i\}$ be any two sequences of numbers such that

$$\psi(x, n) = \prod_{i=1}^{n} (a_i + b_i x) \neq 0 \qquad (1.1)$$

for non negative integers x, n with $\psi(x, 0) = 1$. Then we have the following reciprocal formulas

$$\begin{cases} f(n) = (-\underset{x}{\triangle})_0^n \{g(x)\psi(x, n)\} \\ g(n) = (-\underset{x}{\triangle})_0^n \{f(x)\dfrac{a_{x+1} + xb_{x+1}}{(n, x+1)}\} \end{cases} \qquad (1.2)$$

where $\underset{x}{\triangle}$ denotes the difference operator with unit increment in x and

$$(-\underset{x}{\triangle})_t^n f(x) = \{(-\underset{x}{\triangle})^n f(x)\}_{x=t} \qquad (1.3)$$

As the basic tool this inverse relation has been successfully ussd to construct interpolation formulas (cf.[2—4]). A natural problem is to pursue the multifold analog for Gould Hsu's inversion which is significant of constructing multivariate interpolation formulas. But little progress has been made because of complexity until Hsu [5] discovers a special bivariate invers sion recently. For convenience Hsu's formula be may rewriten in the equivalent form.

Theorem I. Let $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$ be any four sequences of numbers

* Received Nov.5, 1987.

such that

$$\psi(x, y; m, n) = \prod_{i=1}^{m}(a_i + b_i x)\prod_{j=1}^{n}(c_j + d_j x - \frac{y}{j}) \neq 0 \qquad (1.4)$$

for non-negative integers x, y, m and n with the convention $\prod_{i=1}^{0} * = 1$.

Then we have the bivariate reciprocal formulas

$$\begin{cases} f(m, n) = (-\underset{x}{\triangle})_0^m(-\underset{y}{\triangle})_0^n\{g(x, y)\psi(x, y; m, n)\} & (1.5) \\ g(m, n) = (-\underset{x}{\triangle})_0^m(-\underset{y}{\triangle})_0^n\{f(x, y)\dfrac{(a_{x+1} + xb_{x+1})(c_{y+1} + md_{y+1} - \frac{y}{y+1})}{\psi(m, n; x+1, y+1)}\} & (1.6) \end{cases}$$

The object of this paper is to present a slight extension of (1.5—1.6) from four sets to five sets of free parameters. The main conclusion may be stated as follows:

Theorem 2. Let $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$, $\{e_i\}$ be any five sequences of complex numbers such that

$$\varphi(x, y; m, n) = \prod_{i=1}^{m}(a_i + xb_i)\prod_{j=1}^{n}(c_j + xd_j + ye_j) \neq 0 \qquad (1.7)$$

for nonnegative integers x, y, m and n with the convention $\prod_{i=1}^{0} * = 1$ and

$$\varepsilon(x, y; i, j) = (a_{i+1} + ib_{i+1})(c_{j+1} + xd_{j+1} + je_{j+1}) \qquad (1.8)$$

Then we have the following reciprocal formulae:

$$\begin{cases} f(m, n) = (-\underset{x}{\triangle})_0^m(-\underset{y}{\triangle})_0^n\{g(x, y)\varphi(x, y; m, n)\} & (1.9) \\ g(m, n) = (-\underset{x}{\triangle})_0^m(-\underset{y}{\triangle})_0^n\{f(x, y)\dfrac{\varepsilon(m, n; x, y)}{\varphi(m, n; x+1, y+1)}\} & (1.10) \end{cases}$$

It is clear that for $e_j = -\frac{1}{j}$, Theorem 2 reduces to Hsu's formulae. Hence this inversion can be seen as an affirmative answer for Hsu's open problem in the case of taking just one set of free parameters identical to zero.

2. Proof of Theorem 2

According to the condition (1.7), it suffices to show that (1.10) satisfies (1.9). Now the substitution of (1.10) into (1.9) gives

$$(-\underset{x}{\triangle})_0^m(-\underset{y}{\triangle})_0^n\{g(x, y)\varphi(x, y; m, n)\}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\varphi(i, j; m, n)(-\underset{x}{\triangle})_0^i(-\underset{y}{\triangle})_0^j\{f(x, y)\frac{\varepsilon(i, j; x, y)}{\varphi(i, j; x+1, y+1}\}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\varphi(i, j; m, n)\sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{r+s}\binom{i}{r}\binom{j}{s}\frac{\varepsilon(i, j; r, s)}{\varphi(i, j; r+1, s+1)}f(r, s)$$

Since

$$\binom{m}{i}\binom{n}{j}\binom{i}{r}\binom{j}{s}=\binom{m}{r}\binom{n}{s}\binom{m-r}{i-r}\binom{n-s}{j-s}$$

the quiartic sum is equal to

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}f(r,s)\sum_{i=r}^{m}\sum_{j=s}^{n}(-1)^{i+j+r+s}\binom{m-r}{i-r}\binom{n-s}{j-s}\frac{\varphi(i,j;m,n)}{\varphi(i,j;r+1,s+1)}\varepsilon(i,j;r,s)$$

which will reduce to $f(m,n)$ if we can show that

$$\sum_{i=r}^{m}\sum_{j=s}^{n}(-1)^{i+j+r+s}\binom{m-r}{i-r}\binom{n-s}{j-s}\frac{\varphi(i,j;m,n)}{\varphi(i,j;r+1,s+1)}\varepsilon(i,j;r,s)=\delta_{m,r}\delta_{n,s}$$

or equivalently

$$(-\underset{x}{\triangle})_r^{m-r}(-\underset{y}{\triangle})_s^{n-s}\left\{\frac{\varphi(x,y;m,n)}{\varphi(x,y;r+1,s+1)}\varepsilon(x,y;r,s)\right\}=\delta_{m,r}\delta_{n,s} \tag{2.1}$$

Where $\delta_{i,j}$ is Kronecker delta.

Next we verify (2.1) in steps.

a). $r=m,\ s=n$. It is clear thar the left member of (2.1) equals

$$\frac{\varphi(m,n;m,n)}{\varphi(m,n;m+1,n+1)}\varepsilon(m,n;m,n)=1$$

by (1.7—1.8)

b). $r<m,\ s=n$. (1.7—1.8) assert that

$$\frac{\varphi(x,n;m,n)}{\varphi(x,n;r+1,n+1)}\varepsilon(x,y;r,n)=(a_{r+1}+rb_{r+1})\prod_{i=r+2}^{m}(a_i+xb_i).$$

This is a polynomial in x of degree equal to $m-r-1$. Therefore its $(m-r)$ th difference is zero, i.e., the left hand side of (2.1) vanishes in this case.

c). $r=m,\ s<n$. Similar to the situation b), we have

$$\frac{\varphi(m,y;m,n)}{\varphi(m,y;m+1,s+1)}\varepsilon(m,y;m,s)=(c_{s+1}+md_{s+1}+se_{s+1})\prod_{j=s+2}^{n}(c_j+md_j+ye_j).$$

which is a polynomial in y of degree equal to $n-s-1$. Hence the left member of (2.1) vanishes.

d). $r<m,\ s<n$. It is obvious that

$$\frac{\varphi(x,y;m,n)}{\varphi(x,y;r+1,s+1)}\varepsilon(x,y;r,s)$$

is a polynomial in x and y with degree equal to $m+n-r-s-1$. Then its $(m+n-r-s)$ th difference, i.e., the left hand side of (2.1) vanishes.

Hence (2.1) follows from a)—b) and this complete the proof of Theorem 2.

3. Rotated form and special cases

Similar to the method finding a rotated form for a given inversion stated in [1], it is not difficult to obtain the rotated form of Theorem 2 as follows.

Theorem 3. Under the conditions of Theorem 2, the systems of equations

$$\begin{cases} F(m,n)=\sum_{i=m}^{M}\sum_{j=n}^{N}(-1)^{i+j}\binom{i}{m}\binom{j}{n}\varphi(m,n;i,j)G(i,j) \\ G(m,n)=\sum_{i=m}^{M}\sum_{j=n}^{N}(-1)^{i+j}\binom{i}{m}\binom{j}{n}\frac{\varepsilon(i,j;m,n)}{\varphi(i,j;m+1,n+1)}F(i,j) \end{cases} \quad \binom{o\leqslant m\leqslant M}{o\leqslant n\leqslant N} \tag{3.1}$$

are equivalent. Where M, N are fixed positive integers or infinity.

Note that Theorem 2 can be reweiten in a more explicit form

$$\begin{cases} f(m,n)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\varphi(i,j;m,n)g(i,j) \\ g(m,n)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\frac{\varepsilon(m,n;i,j)}{\varphi(m,n;i+1,j+1)}f(i,j) \end{cases} \tag{3.2}$$

For $b_i=b$, $d_i=d$ and $e_i=e$, we take $\{a_i^r|1\leqslant r\leqslant 3\}=\{a,a-i+1,a+i\}$ and $\{c_j^s|1\leqslant s\leqslant 3\}=\{c,c-j+1,c+j\}$ alternatively. Then define the coefficients as follows:

$$\{A_i^r(x);1\leqslant i\leqslant 3\}=\{\frac{(a+bx)^i}{i!},\ \binom{a+bx}{i},\ \binom{a+bx+i}{i}\} \tag{3.3}$$

$$\{B_j^s(x,y);1\leqslant s\leqslant 3\}=\{\frac{(c+dx+ey)^j}{j!},\ \binom{c+dx+ey}{j},\ \binom{c+dx+ey+j}{j}\} \tag{3.4}$$

$$\{u_i^r(x);\ 1\leqslant r\leqslant 3\}=\{(a+bx),\ (a+bx-i),\ (a+bx+i+1)\} \tag{3.5}$$

$$\{V_j^s(x,y);1\leqslant s\leqslant 3\}=\{c+dx+ey,\ c+dx+ey-j,\ c+dx+ey+j+1\} \tag{3.6}$$

Corresponding to the alternate selections of a_i^r and c_j^s, φ and ε sequences can be expressed in terms of above sequences.

$$\varphi_r^s(x,y;i,j)=i!\,j!\,A_i^r(x)B_j^s(x,y) \qquad (1\leqslant r,s\leqslant 3) \tag{3.7}$$

$$\varepsilon_r^s(x,y;i,j)=u_i^r(i)v_j^s(x,j) \qquad (1\leqslant r,s\leqslant 3) \tag{3.8}$$

It is obvious that φ satisfies the following recurrence relation

$$\varphi_r^s(x,y;i+1,j+1)=u_i^r(x)v_j^s(x,y)\varphi_r^s(x,y;i,j) \quad (1\leqslant r,s\leqslant 3) \tag{3.9}$$

Relacing $f(m,n)$ by $F(m,n)m!\,n!$ in (3.2) we establish the following nine explicit inverse relations by means of (3.7—3.9).

Theorem 4. Let A, B and U, V be defined by (3.3—3.6). Then the following nine pairs of systems are reciprocal

$$\begin{cases} F(m,n)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{j}\binom{n}{j}A_m^r(i)B_n^s(i,j)G(i,j) \\ G(m,n)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\frac{U_i^r(i)V_j^s(m,j)}{U_i^r(m)V_j^s(m,n)}\cdot\frac{F(i,j)}{A_i^r(m)B_j^s(m,n)} \end{cases}$$

for $1\leqslant r,s\leqslant 3$.

These inverse formulas may be seen as the bivariate analogs of inverse relations listed in [1].

4. Applications to interpolation formulae

In the similar manner to Hsu [5], Theorem 2 can be used to construct interpolation formula as follows:

Theorem5. Given a function $f(x, y)$ which is well defined for non-negative integers x and y, we define a formal bivariate series in the form

$$S(f; x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{x}{i}\binom{y}{j} \frac{\varepsilon(x, y; i, j)}{\varphi(x, y; i+1, j+1)} \left(\frac{\triangle}{x}\right)_0^i \left(\frac{\triangle}{y}\right)_0^j \{f(x, y)\varphi(x, y; i, j)\}. \tag{4.1}$$

Then for all non-negative integers m and n

$$S(f; m, n) = f(m, n) \tag{4.2}$$

follows from Theorem 2.Hence $S(f; x, y)$ is a rational interpolation series of $f(x, y)$.

When taking d_j identical to zero $S(f;x,y)$ reduces to the tensor product form of one dimensional case developed in [2—4].

For some particular setting of free parameters (e.g., as section 3), (4.1) can yields various concrete bivariate rational interpolation formulae. Among them there may exist some efficient ones for numerical analysis.

References

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