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A Note on Hamiltonian Grids*

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Abstract

A complete grid $G_{m,n}$ is the cartesian product of two paths P_m and P_n. In this paper, it is proved that a class of complete grids with two vertices removed are hamiltonian. This result settles a conjecture of S.M. Hedetniemi, S.T. Hedetniemi and P.J. Slater in positive.

I. Introduction

Let $m\geqslant 2$ and $n\geqslant 2$ be two integers. The following concepts were defined in [1]: A complete grid $G_{m,n}$ is a graph having mn vertices which are connected to form a rectangular lattice in the plane, i.e., all edges of $G_{m,n}$ connect vertices along horizontal or vertical lines. A grid is a subgraph of a complete grid. Obviously a complete grid $G_{m,n}$ is isomorphic to the cartesian product of two paths p_m and p_n.

Let $G_{m,n}$ be a complete grid with $V(G_{m,n})=\{V_{i,j}|1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$. Let $V_1=\{V_{i,j}|i+j \text{ is even}\}$, $V_2=\{V_{i,j}|i+j \text{ is odd}\}$.

In this paper, we shall disucss the hamiltonian property of grids.

Theorem A([2]): $G_{m,n}$ is hamiltonian iff mn is even.

Theorem B([1]): $G_{2r+1,2s+1}-\{v\}$ is hamiltonian iff $v\in V_1$.

Theorem C([1]): Let $G_{2r,2s}$ be a complete grid where $r,s\geqslant 2$, $S=\{u,v\}$ where $u\in V_1$, $v\in V_2$. Then $G_{2r,2s}-S$ is hamiltonian iff $G_{2r,2s}-S$ is 2-connected.

Theorem D([1]): Let $m\geqslant 4$ be any integer, $S=\{u,v\}$ where $u\in V_1$, $v\in V_2$. The $G_{m,4}-S$ is hamiltonian iff $G_{m,4}-S$ is 2-connected.

Theorem E([1]): Let $n\geqslant 4$ be even, $S=\{u,v\}$ where $u\in V_1$. and $v\in V_2$. Then $G_{7,n}-S$ is hamiltonian iff $G_{7,n}-S$ is 2-connected.

When $m=3$ or 5, 2-connectivity of $G_{m,n}-\{u,v\}$ can not imply that $G_{m,n}-\{u,v\}$ is hamiltonian, where $n\geqslant 6$ is even, $u\in V_1$ and $v\in V_2$ In these two cases, S.M. Hedetniemi and S.T. Hedetniemi and P.J. Slater ([1]) settled the problem which grids $G_{m,n}-\{u,v\}$ are hamiltonian. In general case, they posed the follo-

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wing conjecture.

Conjecture 1([1]): Let r,s be two positive integers, $r\geqslant 3$, $u\in V_1$ and $v\in V_2$. Then every 2-connected grid $G_{2r+1,2s}-\{u,v\}$ is hamiltonian.

We shall prove this conjecture in next section.

2. Main Result

In this section, Conjecture 1 is proved.

Theorem: Let $m\geqslant 7$ be odd, $n\geqslant 4$ even and $G_{m,n}$ a complete grid. If $u\in V_1$, $v\in V_2$, and $G_{m,n}-\{u,v\}$ is 2-connected, then $G_{m,n}-\{u,v\}$ is hamiltonian.

Proof: Let $S=\{u,v\}$, where $u\in V_1$, $v\in V_2$, and $G_{m,n}-S$ 2-connected. Denote $m=2r+1$. $n=2s$. We prove the theorem by induction on $r+s$. If $r=3$ or $s=2$, then the theorem is valid by Theorem E and Theorem D. Thus we assume that $r+s>5$, and $r>3$, $s>2$.

Let $J_i=\{v_{i,1},v_{i,2},\cdots,v_{i,n}\}$, $L_j=\{v_{1,j},v_{2,j},\cdots,v_{m,j}\}$, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant n$. If $S\cap(J_{m-2}\cup J_{m-1}\cup J_m\cup\{v_{m-3,1},v_{m-3,n}\})=\phi$, then the 2-connectivity of $G_{m,n}-S$ implies that $G_{m,n}-(S\cup J_{m-1}\cup J_m)$ is 2-connected. The latter is hamiltonian by the induction hypothesis, so does the former. Therefore, we can assume that $S\cap(J_1\cup J_2\cup J_3\cup\{v_{4,1},v_{4,n}\})\neq\phi\neq S\cap(J_{m-2}\cup J_{m-1}\cup J_m\cup\{v_{m-3,1},v_{m-3,n}\})$, and $S\cap(L_1\cup L_2\cup L_3\cup\{v_{1,4},v_{m,4}\})\neq\phi\neq S\cap(L_{n-2}\cup L_{n-1}\cup L_n\cup\{v_{1,n-3},v_{m,n-3}\})$. Since $|S|=2$, we can obtain that either $S\subset X_1\cup X_2$ or $S\subset Y_1\cup Y_2$, where $X_1=\{v_{1,1},v_{1,3},v_{2,2},v_{2,3},v_{3,1},v_{3,2},v_{3,3},v_{4,1},v_{1,4}\}$, $X_2=\{v_{m-3,n},v_{m-2,n-2},v_{m-2,n-1},v_{m-2,n},v_{m-1,n-2},v_{m-1,n-1},v_{m,n-3},v_{m,n-2},v_{m,n}\}$, $Y_1=\{v_{m-3,1},v_{m-2,1},v_{m-2,2},v_{m-2,3},v_{m-1,2},v_{m-1,3},v_{m,1},v_{m,3},v_{m,4}\}$, $Y_2=\{v_{1,n-3},v_{1,n-2},v_{1,n},v_{2,n-2},v_{2,n},v_{3,n-2},v_{3,n-1},v_{3,n},v_{4,n}\}$. Without loss of generality, we suppose that $S\subset X_1\cup X_2$, and $S\cap X_1\neq\phi\neq S\cap X_2$.

If $v=v_{m-2,n}$, then $G_{m,n}-(S\cup J_{m-1}\cup J_m)$ is 2-connected. Thus $G_{m,n}-(S\cup J_{m-1}\cup J_m)$ is hamiltonian, so does $G_{m,n}-S$. Similarly, we can get that $G_{m,n}-S$ is hamiltonian if $S\cap\{v_{1,3},v_{3,1},v_{3,3},v_{m-2,n-2},v_{m,n-2}\}\neq\phi$.

If $u\in\{v_{1,1},v_{2,2}\}\subset V_1$, then $v\in\{v_{m-1,n-1},v_{m,n}\}\subset V_2$. Since both $\langle L_1\cup L_2\cup L_3\rangle-\{u\}$ and $\langle L_4\cup\cdots\cup L_n\rangle-\{v\}$ are hamilionian, $G_{m,n}-S$ is hamiltonian.

Let $v\in\{v_{1,4},v_{2,3},v_{3,2},v_{4,1}\}\subset V_2$, $u\in\{v_{m-3,n},v_{m-2,n-1},v_{m-1,n-2},v_{m,n-3}\}\subset V_1$. There are 16 cases to be considered.

$S_1=\{v_{1,4}v_{m-3,n}\}$, $S_2=\{v_{1,4},v_{m-2,n-1}\}$,
$S_3=\{v_{1,4},v_{m-1,n-2}\}$, $S_4=\{v_{1,1},v_{m,n-3}\}$,
$S_5=\{v_{2,3},v_{m-3,n}\}$, $S_6=\{v_{2,3},v_{m-2,n-1}\}$,
$S_7=\{v_{2,3},v_{m-1,n-2}\}$, $S_8=\{v_{2,3},v_{m,n-3}\}$,
$S_9=\{v_{3,2},v_{m-3,n}\}$, $S_{10}=\{v_{3,2},v_{m-2,n-1}\}$,
$S_{11}=\{v_{3,2},v_{m-1,n-2}\}$, $S_{12}=\{v_{3,2},v_{m,n-3}\}$,
$S_{13}=\{v_{4,1},v_{m-3,n}\}$, $S_{14}=\{v_{4,1},v_{m-2,n-1}\}$,
$S_{15}=\{v_{4,1},v_{m-1,n-2}\}$, $S_{16}=\{v_{4,1},v_{m,n-3}\}$.

Let $H = G_{m,n} - (J_1 \cup J_2 \cup J_{m-1} \cup J_m \cup L_1 \cup L_2 \cup L_{n-1} \cup L_n)$. Then H is a complete grid of $(m-4)\times(n-4)$-order, thus, H is hamiltonian. Let P_1 be a hamiltonian path of H:

$$P_1 = v_{4,3}v_{3,3}v_{3,4}v_{4,4}v_{5,4}\cdots v_{m-3,4}v_{m-3,5}v_{m-4,5}v_{m-5,5}\cdots v_{3,5}v_{3,6}v_{4,6}v_{5,6}\cdots v_{3,n-2}v_{4,n-2}\cdots v_{m-3,n-2}v_{m-2,n-2}v_{m-2,n-3}v_{m-2,n-4}\cdots v_{m-2,4}v_{m-2,3}v_{m-3,3}\cdots v_{5,3}.$$

Let

$$P_2 = v_{5,2}v_{5,1}v_{6,1}v_{6,2}\cdots v_{m-2,1}v_{m-1,1}v_{m,1}v_{m,2}v_{m-1,2}v_{m-1,3}v_{m,3}\cdots v_{m,n-4}v_{m-1,n-4}v_{m-1,n-3},$$

and

$$P_3 = v_{m-3,n-1}v_{m-4,n-1}v_{m-4,n}\cdots v_{4,n}v_{4,n-1}v_{3,n-1}v_{3,n}v_{2,n}v_{1,n}v_{1,n-1}v_{2,n-1}\cdots v_{2,6}v_{1,6}v_{1,5}v_{2,5}v_{2,4}.$$

Then P_2 and P_3 are two disjoint paths of the subgraph $\langle J_1 \cup J_2 \cup J_{m-1} \cup J_m \cup L_1 \cup L_2 \cup L_{n-1} \cup L_n \rangle$ of $G_{m,n}$.

By means of P_1, P_2 and P_3, we can easily construct the hamiltonian cycles in $G_{m,n} - S_i$, where $i = 1, 2, \cdots, 16$. As examples, we only consider the first two cases. The other cases are completely similar.

Case 1. $v_{2,4}v_{2,3}v_{1,3}v_{1,2}v_{1,1}v_{2,1}v_{2,2}v_{3,2}v_{3,1}v_{4,1}v_{4,2}P_1P_2v_{m,n-3}v_{m,n-2}v_{m-1,n-2}v_{m-1,n-1}v_{m,n-1}v_{m,n}v_{m-1,n}v_{m-2,n}v_{m-2,n-1}P_3$ is a hamiltonian cycle in $G_{m,n} - S_1$.

Case 2. $v_{2,4}v_{2,3}v_{1,3}v_{1,2}v_{1,1}v_{2,1}v_{2,2}v_{3,2}v_{3,1}v_{4,1}v_{4,2}P_1P_2v_{m,n-3}v_{m,n-2}v_{m-1,n-2}v_{m-1,n-1}v_{m,n-1}v_{m,n}v_{m-1,n}v_{m-2,n}v_{m-3,n}P_3$ is a hamiltonian cycle in $G_{m,n} - S_2$.

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References

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