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Carleson Measure and Multipliers on Bergman Space*

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1. Introduction. In 1962, Carleson [2] proved the well known Carleson measure theorem for Hardy space. In 1975. Hastings [5] obtained an analogous theorem for Bergman space. In this paper we prove a Carleson type theorem for the quasi-normal weighted Bergman space using a way other than Hastings'. As an application we obtain a theorem on multiplier theory. This generalizes and includes Attele's results [1] and Taylor's results [6].

Let φ be positive continuous function on $[0, 1)$. It will be said that φ is quasi-normal if there exists $a>0$ such that $\varphi(r)/(1-r)^a$ is increasing on $[0, 1)$ and $\frac{1}{1-r}\int_r^1 \varphi(\rho)\,d\rho = 0(\varphi(r))$ for $r\geq 0$.

If φ is quasi-normal and $0<p\leq\infty$, a function $f(z)$ analytic in $|z|<1$ is said to belong to the quasi-normal weighted Bergman space A^p_φ if

$$\|f\|_{p,\varphi} = \left(\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\varphi(r)\,|f(re^{i\theta})|^p d\theta dr\right)^{1/p}<\infty.$$

In particular, $A^p_1 = A^p$ is called the Bergman space. A real-valued function $u(z)$ harmonic in $|z|<1$ is said to be of space a^p_φ if $\|u\|_{p,\varphi}<\infty$

2. The Carleson type measure theorem.

Theorem 1. Let μ be a finite measure on $|z|<1$, and let φ be quasi-normal, $1<p\leq q<\infty$. Then there exisis a constant $c>0$ such that

$$\left(\iint_{|z|<1} f^q(z)\,d\mu(z)\right)^{1/q}\leq c\,\|f\|_{p,\varphi} \tag{1}$$

for every positive subharmonic function f in $|z|<1$, if and only if

$$\mu(S(I)) = 0(|I|^{2q/p}\varphi(1-|I|)^{q/p}) \tag{2}$$

for every set $S(I)$ of the form

$$S(I) = \left\{ z: \frac{z}{|z|}\in I,\ 1-|I|<|z|<1\right\} \tag{3}$$

where I is a subarc on the unit circle and $|I|$ denotes the normalized arc

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length of I.

Proof. If (1) holds. Let $S(I)$ be a set of the form (3) and pick w centered in $S(I)$. It is easily observed that $1-|w|$ and $|1-\tilde{w}z|$ for z in $S(I)$ are both comparable to $|I|$. Let

$$f(z)=|1-\tilde{w}z|^{-(3+a)/p}$$

Then $$\|f\|_{p,\varphi}^{p}\leqslant c_1\int_0^1\frac{\varphi(r)}{(1-|w|r)^{2+a}}\,\mathrm{d}r$$

$$\leqslant c_1\frac{\varphi(|w|)}{(1-|w|)^{a}}\int_0^{|w|}\frac{\mathrm{d}r}{(1-|w|r)^{2}}+c_1\frac{1}{(1-|w|)^{2+a}}\int_{|w|}^1\varphi(r)\mathrm{d}r=O(|I|^{-1-a}\varphi(1-|I|/2))$$

But $f^p\geqslant c_2|I|^{-(3+a)}$ follows from $|1-\tilde{w}z|\sim|I|$ for z in $S(I)$. Therefore $\mu(S(I))=O(|I|^{2q/p}\varphi(1-|I|/2)^{q/p})$. Since φ is quasi normal,

$$\varphi(1-|I|/2)=\frac{\varphi(1-|I|/2)}{(|I|/2)^{a}}\left(\frac{|I|}{2}\right)^{a}\leqslant\frac{2(1+a)}{|I|}\int_{1-\frac{|I|}{2}}^{1}\frac{\varphi(\rho)}{(1-\rho)^{a}}(1-\rho)^{a}\mathrm{d}\rho$$

$$\leqslant\frac{2(1+a)}{|I|}\int_{1-|I|}^{1}\varphi(\rho)\mathrm{d}\rho=O(\varphi(1-|I|)). \tag{4}$$

Hence (2) holds.

Conversely, if (2) holds for every set $S(I)$ of the form (3). We need only to show that (1) holds for every positive subharmonic function f in $|z|<1$ satisfying $\|f\|_{p,\varphi}<\infty$. Let m_φ be the measure $\varphi(|z|)\mathrm{d}x\mathrm{d}y$ on $|z|<1$. For $u\in L^1(\mathrm{d}m_\varphi)$,

put $$M_\varphi[u](z)=\sup_I\frac{1}{m_\varphi(S(I))}\iint_{S(I)}|u|\mathrm{d}m_\varphi,$$

where the supremum is taken over all $I\supset I_z$, here I_z is the arc centered at $z/|z|$ and $|I_z|=1-|z|$. It is easily known that there is a constant C such that $\mu(S(I))\leqslant Cm_\varphi(S(I))^{q/p}$. Now it follows that the weak type $(1,q/p)$ condition $\mu\{M_\varphi[u]>S\}\leqslant C(S^{-1}\int|u|\mathrm{d}m_\varphi)^{q/p}$ holds for some constant C. See Duren's book [3, Chap. 9.5] for details of a similar argument. Since M_φ is obviously a sublinear operator of type (∞,∞), we use the Marcinkiewicz interpolation theorem [8, Chap. XII] to conclude that M_φ is a bounded sublinear operator mapping $L^p(m_\varphi)$ into $L^q(\mu)$. Now we assume that M_φ has the maximal property: there exists a constant $C>0$ such that

$$|u(z)|\leqslant CM_\varphi[u](z). \tag{5}$$

Then $$\left(\iint_{|Z|<1}f^q\mathrm{d}\mu\right)^{1/q}\leqslant C\left(\iint_{|Z|<1}M_\varphi[f]^q\mathrm{d}\mu\right)^{1/q}\leqslant C\|M_\varphi\|\,\|f\|_{p,\varphi}.$$

So (1) holds with $c=C\|M_\varphi\|$. Now we prove that M_φ has the desired property. Fix $|w|<1$ and let D_w denote the disc centered at w with radius $(1-|w|)/2$. Let I be the subarc centered at $w/|w|$ and such that $|I|=2(1-|w|)$ or 1 whichever is smaller. For z in D_w, we have

$$\varphi(|z|)\geqslant C\frac{1}{1-|z|}\int_{|z|}^{1}\varphi(r)\mathrm{d}r\geqslant\frac{2C}{3(1-|w|)}\int_{\frac{1+|w|}{2}}^{1}\varphi(r)\mathrm{d}r$$

$$\geqslant\frac{C}{3(1+a)}\varphi\left(\frac{1+|w|}{2}\right)\geqslant\frac{C}{3(1+a)}2^{-a}\varphi(|w|).$$

Then

$$|u(w)|\leqslant C_1(1-|w|)^{-2}\iint_{D_w}|u|\mathrm{d}x\mathrm{d}y\leqslant C_2(1-|w|)^{-2}\varphi(|w|)^{-1}\iint_{D_w}|u|\varphi(|z|)\mathrm{d}x\mathrm{d}y$$

$$\leqslant Cm_\varphi(S(I))^{-1}\iint_{S(I)}|u|\mathrm{d}m_\varphi\leqslant CM_\varphi[u](w).$$

A corollary follows immediately.

Corollary 1. Let μ be a finite measure on $|z|<1$, and let φ be quasi-normal and $0<p\leqslant q<\infty$. Then there exists a constant $C>0$ such that

$$\left(\iint_{|Z|<1}|u(z)|^q\mathrm{d}\mu(z)\right)^{1/q}\leqslant C\|u\|_{p,\varphi} \tag{6}$$

for all $u\in a_\varphi^p$, if and only if (2) holds.

Proof. First we note that if u is in A_φ^p, then $|u|^{p/2}$ is positive subharmonic, we use Theorem 1 to conclude that (6) holds for all u is in A_φ^p if and only if (2) holds. (6) implying (2) is same as the proof of Theorem 1. Now suppose (2) holds. For $u\in a_\varphi^p$, let $\tilde{u}$ be the harmonic conjugate of u. By Theorem 3 of [7] $f=u+\tilde{u}i\in A_\varphi^p$, and there exists a constant $C_1>0$ such that $\|f\|_{p,\varphi}\leqslant C_1\cdot\|u\|_{p,\varphi}$. Hence

$$\left(\iint_{|Z|<1}|u(z)|^q\mathrm{d}\mu(z)\right)^{1/q}\leqslant\left(\iint_{|Z|<1}|f(z)|^q\mathrm{d}\mu(z)\right)^{1/q}\leqslant C\|f\|_{p,\varphi}\leqslant CC_1\|u\|_{p,\varphi}.$$

Thus (6) holds and completing the proof.

Although we have worked in the unit disc, we can extend our results to the unit polydisc U^n. Let σ_n be $2n$-dimensional normalized Lebesgue volume measure restricted to U^n. Suppose $0<p\leqslant\infty$, $\varphi(|z|)=\prod_{j=1}^{n}\varphi_j(|z_j|)$, $\varphi_j(|z_j|)$ is quasi-normal, $1\leqslant j\leqslant n$, a function $f(z)$ analytic in U^n is said to belong to the quasi-normal weighted Bergman space $A_\varphi^p(U^n)$ if

$$\|f\|_{p,\varphi}=\left(\int_{U^n}\varphi(|z|)|f(z)|^p\mathrm{d}\sigma_n(z)\right)^{1/p}<\infty,$$

a real-valued function $u(z)$ harmonic in U^n is said to be of space $u_\varphi^p(U^n)$ if $\|u\|_{p,\varphi}<\infty$.

Theorem 2. Let μ be a finite measure on U^n, and let φ be quasi-normal and $1<p\leqslant q<\infty$. Then there exists a constant $C>0$ such that

$$\left(\int_{U^n}f^q(z)\mathrm{d}\mu(z)\right)^{1/p}\leqslant C\|f\|_{p,\varphi} \tag{7}$$

for all positive n-subharmonic function f in U^n if and only if

$$\mu(S(I))=O\left(\prod_{j=1}^{n}|I_j|^{2q/p}\varphi_j(1-|I_j|)^{q/p}\right) \tag{8}$$

for every set $\mathrm{S}(I)$ of the form

$$\mathrm{S}(I)=(\mathrm{S}(I_1),\cdots,\mathrm{S}(I_n)), \tag{9}$$

where $\mathrm{S}(I_j)$ is a set of the form (3), $j=1,\cdots,n$.

Since the proof of the theorem follows from repeated application of the proof of the theorem 1, we omit that.

Same as Corollary 1, Theorem 2 also holds for $0<p\leq q<\infty$ if we replace the space of positive n subharmonic by $A^p_\varphi(U^n)$ or $a^p_\varphi(U^n)$.

3. Multipliers. If F and G are two arbitrary collections of functions, we let $M(F,G)$ be the collection of all functions which multiply, F into G. i.e., fg is in G for all g in F. If $f\in M(F,G)$, the multiplication operator $\mathrm{M}_f: F\to G$ is defined by $\mathrm{M}_f(g)=fg$ for $g\in F$.

Lemma 1. Suppose $0<p\leq q<\infty$, φ and ψ are quasi normal. Then $f\in\mathrm{M}(a^q_\varphi, a^q_\psi)$ if and only if f is harmonic and $|f|^q\psi(|z|)\mathrm{d}x\mathrm{d}y$ satisfies (2).

Proof. If f is in $\mathrm{M}(a^p_\varphi, a^q_\psi)$, Lemma 3.7 of Chapter in [4] shows that the linear functionals of evaluation at a point are continuous and thus an application of the Closed Graph Theorem shows that M_f is bounded. Combining this and Corollary 1, the lemma is obtained.

Theorem 3. Suppose $0<p\leq q<\infty$, φ and ψ are quasi normal. A harmonic function f belongs to $\mathrm{M}(a^p_\varphi, a^q_\psi)$ if and only if

$$f(z)=O((1-|z|)^{2/p-2/q}\varphi(|z|)^{1/p}\psi(|z|)^{-1/q}).$$

Proof. If $f(z)=O((1-|z|)^{2/p-2/q}\varphi(|z|)^{1/p}\psi(|z|)^{-1/q})$. For any set $\mathrm{S}(I)$ of the form (3), it is easily shown that

$$\iint_{\mathrm{S}(I)}\psi(|z|)|f(z)|^q\mathrm{d}x\mathrm{d}y=O(|I|^{2q/p}\varphi(1-|I|)^{q/p}), \tag{10}$$

It follows that $f\in\mathrm{M}(a^p_\varphi, a^q_\psi)$. Conversely, if $f\in\mathrm{M}(a^p_\varphi, a^q_\psi)$. Same as the proof of (5), we have

$$f(z)=O((1-|z|)^{-2}\psi(|z|)^{-1}\iint_{\mathrm{S}(I)}|f(w)|^q\psi(|w|)\mathrm{d}u\mathrm{d}v)^{1/q})$$

Then (10) implies $f(z)=O((1-|z|)^{2/p-2/q}\varphi(|z|)^{1/p}\psi(|z|)^{-1/q}$. This prove the theorem.

Same as the proof Theorem 3, we can obtain the following theorem.

Theorem 3′. Suppose $0<p\leq q<\infty$, φ and ψ are quasi normal. An analytic function f belongs to $\mathrm{M}(\mathrm{A}^p_\varphi, \mathrm{A}^q_\psi)$ if and only if $f(z)=O((1-|z|)^{2/p-2/q}\varphi(|z|)^{1/p}\psi(|z|)^{1/q})$.

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$$g_{\beta,\gamma}(x) = \begin{cases} |x|^{\beta}|\ln^{\gamma} x|, & \text{if } r \text{ is even}, \\ x|x|^{\beta-1}|\ln^{\gamma} x|, & \text{if } r \text{ is odd}, \end{cases} \quad 0<a<1, \gamma>0,$$

then

$$E_n(g_{\beta,\gamma}) \sim n^{-2\beta}\ln^{\gamma} n.$$

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