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A Commutativity Condition on Semiprime Rings*

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Let R be an associative ring, $[a,b]$ the commutator of a,b.i.e. $[a,b]=ab-ba$ and $Z(R)$ the center of R.

For a semiprime ring R, the commutative conditions studied by many authors ([1]-[5]), are as follows $(xy)^n-x^sy^t\in Z(R)$. This paper will keep on studying the conditions above and obtain the following result

Theorem Let R be a semiprime ring. R is a commutative ring if and only if the following condition holds

For any $x,y\in R$, there exist integers $n=n(x)>1, s=s(x)>1$ and $t=t(x)>1$ (or $n=n(y)>1$, $s=s(y)>1$ and $t=t(y)>1$) such that

$$(xy)^n-x^sy^t\in Z(R). \tag{1}$$

Lemma 1 A nonzero one-sided nil ideal with finite upper-index(i.e. supremum of all nil index of elements in the ideal.) contains at lest a nonzero one-sided nilpotent ideal.

For a proof see [6].

Lemma 2 A ring, which has no nilpotent elements, is isomorphic to a meta-direct sum of some rings which has no null divisors.

For a proof see [7].

Lemma 3 Let R be a ring which may be embeded in a division ring. If R satisfies the following condition

For any $x,y\in R$, there exist positive integers $s=s(x)$, $t=t(x)$ (or $s=s(y)$, $t=t(y)$) such that

$$[x^sy^t, yx]=0 \tag{2}$$

Then R must be a commutative ring.

Proof Let $x\neq 0$ $(x\in R)$. For any $y\in R$ with $y\neq 0$, the condition (2) implies there exist positive integers $s=s(x)$ and $t=t(x)$ such that

$$x^sy^{t+1}x=yx^{s+1}y^t \tag{3}$$

and

$$x^sy^{2(t+1)}x=y^2x^{s+1}y^{2t}. \tag{4}$$

Since

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$$y^2x^{s+1}y^{2t} = y(yx^{s+1}y^t)y^t = y(x^sy^{t+1}x)y^t$$
$$= (yx^{s+1}y^t)y^{-t}x^{-1}y^{t+1}xy^t$$
$$= x^sy^{t+1}xy^{-t}x^{-1}y^{t+1}xy^t. \qquad (5)$$

Subsituting the (5) into (4), we get

$$y^{t+1}x = xy^{-t}x^{-1}y^{t+1}xy^t.$$

Hence,

$$y^tx^{-1}y^{t+1}x = x^{-1}y^{t+1}xy^t. \qquad (6)$$

So y^t is commutative with $x^{-1}y^{t+1}$.

If $x+y\neq 0$, writting $x+y$ instead of x above, we obtain the following expression analogy to (6)

$$y^l(x+y)^{-1}y^{l+1}(x+y) = (x+y)^{-1}y^{l+1}(x+y)y^l, \qquad (7)$$

where $l=l(x+y)$ is a positive integer.

Writting $k=(t+1)(l+1)$ and setting

$$z_1 = x^{-1}y^kx, \quad z_2 = (x+y)^{-1}y^k(x+y). \qquad (8)$$

By (6), (7), it is easy to see that both z_1 and z_2 are commutative with y^{tl}. Because of (8) we have

$$y^kx = xz_1, \quad y^k(x+y) = (x+y)z_2.$$

Thus

$$y^{k+1} - yz_2 = x(z_2 - z_1). \qquad (9)$$

Since the left side of (9) is commutative with y^{tl} and so is the right side of (9). Hence, $(y^{tl}x - xy^{tl})(z_2 - z_1) = 0$. Since R has no null divisors, then $y^{tl}x - xy^{tl} = 0$, or $z_2 - z_1 = 0$. For the first equality, we have $y^{tl}x = xy^{tl}$, For the next equality, in virtue of (9) we get, $y^k = z_2 = z_1 = x^{-1}y^kx$. Thus $xy^k = y^kx$. By [8], we obtain that R is commutative ring.

For the case $s=s(y)$ and $t=t(y)$, the proof is analogous. The proof of lemma is completed.

The Proof of Theorem If R is a commutative ring. It is obvious that the condition (1) holds. On the other hand, if the condition (1) is satisfied, then we will show R is commutative ring.

Let $x\in R$ and $x^2=0$. For any $r\in R$, by condition (1), there exist integers $n=n(x)>1$, $s=s(x)>1$ and $t=t(x)>1$ such that $(xr)^n - x^sr^t = (xr)^n \in Z(R)$. So, $(xr)^nx = x(xr)^n = 0$. Furthermore, $(xr)^{n+1} = 0$. Therefore, xR is an one sided nil ideal of R which upper index is bounded. Since R is a semiprime ring. By lemma 1, we get $xR=(0)$. Hence $x=0$.

Thus, R is a ring has no untrivial nilpotent elements. By lemma 2, we can assume that R has no null divisors.

For any $x, y\in R$, by the condition (1), we have $(xy)^n - x^sy^t \in Z(R)$, where $n=n(x)>1$, $s=s(x)>1$ and $t=t(x)>1$ are all integers.

Hence, for any $r \in R$, we get $[(xy)^n - x^s y^t, r] = 0$. Setting $r = xy$, we have $[x^s y^t, xy] = 0$. That is $x^s y^t xy = xyx^s y^t$. So $x^{s-1} y^t x = yx^s y^{t-1}$, hence $[x^s y^{t-1}, yx] = 0$. It follows that R satisfies Ore condition and may be embeded in a division ring. By lemma 3, R is commutative ring.

In a similar way, we can prove that the case $n = n(y) > 1$, $s = s(y) > 1$ and $t = t(y) > 1$.

The proof of Theorem is completed.

Corollary Let R be a semiprime ring. R is a commutative ring if and only if the following condition holds

For any $x, y \in R$, there exists an integer $n = n(x) > 1$ (or $n = n(y) > 1$) such that $(xy)^n - x^n y^n \in Z(R)$.

Remark The corollary above is a result of theorem 2 in [3].

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半质环的一个交换性条件

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摘 要

定理 设R是半质环，则R是交换环的充分必要条件是：

对任意$x, y \in R$，存在整数$n = n(x) > 1$，$s = s(x) > 1$及$t = t(x) > 1$（或者$n = n(y) > 1$，$s = s(y) > 1$及$t = t(y) > 1$）使得

$$(xy)^n - x^s y^t \in Z(R).$$

其中$Z(R)$是R的中心.