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The Partial Stability for Linear Discrete Systems *

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Abstract

In this paper, we study the partial stability of linear discrete systems by means of Liapunov's functions of quadratic form. We obtain a necessary and sufficient condition for the system being stable with respect to part of variables and generalize Liapunov's equation to the partial stability of linear discrete systems. A method of constructing Liapunov's function of quadratic form for the stability of the systems is given.

§ 1 Introduction

For the study of partial stability and partial asymptotic stability of linear continuous systems, there appeared lot of papers at home and abroad [1-3] having made the problems done. In recent years, lots of authors have paid a great attention to the study of linear discrete systems [4-5]. However, there is no more papers studying the partial stability of linear discrete systems. In this paper, we study the partial stability and partial asymptotic stability by reducing the partial stability to that of all variables.

§ 2 The Partial Asymptotic Stability

Consider the partial asymptotic stability for linear discrete system with respect to $x_1, \cdots, x_m$.

$$x(\tau+1) = A_0 x(\tau) \tag{2.1}$$

where $x(\tau) = (x_1(\tau), \cdots, x_n(\tau))^T \in R^n$, $\tau \in J = \{\tau_0, \tau_0+1, \cdots, \tau_0+k, \cdots\}$, A_0 is an $n\times n$ constant matrix.

If we denote $y(\tau) = (x_1(\tau), \cdots, x_m(\tau))^T$, $z(\tau) = (x_{m+1}(\tau), \cdots, x_n(\tau))^T$. $(p = n-m > 0)$, then system (2.1) is rewritten as (2.2) as follows

$$\begin{cases} y(\tau+1) = Ay(\tau) + Bz(\tau) \\ z(\tau+1) = Cy(\tau) + Dz(\tau) \end{cases} \tag{2.2}$$

where A and B are $m\times m$ and $m\times p$ matrices, C and D are $p\times m$ and $p\times p$

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matrices respectively.

Let $K_s = (B^T, D^T B^T, \cdots, (D^T)^{(s-1)} B^T), s = 1, \cdots, p$, rank $K_p = h \leqslant p$, and $L = \begin{pmatrix} E_m & \\ & L_0 \end{pmatrix}$, $L_0 = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\begin{matrix} h \\ p-h \end{matrix}$, where L_1 is the transposed matrix from h linear independent columns in K_p, L_2 is the orthogonal matrix of L_1, that means $L_1 L_2^T = 0$ and det $L_0 \neq 0$, then system (2.2) is reduced to (2.3) by a nonsingular transformation $U = Lx$

$$\begin{cases} u(\tau+1) = A_1 u(\tau) \\ \theta(\tau+1) = B_1 u(\tau) + D_1 \theta(\tau) \end{cases} \tag{2.3}$$

where $U = (u^T, \theta^T)^T$, $u = (x_1, \cdots, x_m, u_1, \cdots, u_h)^T$, $\theta = (\theta_1, \cdots, \theta_{p-h})^T A_1$, B_1 and D_1 are corresponding matrices.

System $u(\tau+1) = A_1 u(\tau)$ in (2.3) is called the μ-type system from system (2.2).

Theorem 2.1. The necessary and sufficient condition of system (2.2) being asymptotically y-stable is that all the norms of the characteristic roots of A_1 are less than 1.

Proof. We are only to prove the necessity.

Suppose that there exists a characteristic root λ of A_1 with norm $|\lambda| \geqslant 1$, then $\lambda^n \xi (\bar{\xi} = (\xi_1, \xi_2, \cdots, \xi_{m+h})^T$ is a solution of μ type, λ is also a characteristic root of system (2.3) that is $\lambda^n \xi$ $\xi = (\xi_1, \cdots, \xi_{m+h}, \xi_{m+h+1}, \cdots, \xi_n)^T$ is a solution of (2.3). It is easily seen that $\lambda^n \eta (\eta = (\xi_1 \cdots, \xi_m, \eta_1, \cdots, \eta_p)^T)$ is a solution of system (2.2). From that system (2.2) is asymptotically y-stable, we can see that $\xi_1 = \cdots = \xi_m = 0$. Substituting $\lambda^n \eta$ to system (2.2) we have $B(\eta_1, \cdots, \eta_p)^T = 0$.

In the same way, we can prove that

$$BD(\eta_1, \cdots, \eta_p)^T = 0, \cdots, \quad BD^{(p-1)}(\eta_1, \cdots, \eta_p)^T = 0,$$

that is $\xi_{m+1} = \cdots = \xi_{m+h} = 0$. It is in controdiction with the hypothesis $\bar{\xi} \neq 0$.

This completes the proof of the theorem.

Theorem 2.2 Linear discrete system (2.1) is asymptotically y-stable if and only if there exists a y-positive semi definite quadratic form $V(x) = x^T R x$ such that $\Delta V(x)|_{(2.1)} = W(x) = -x^T Q x$ is y negative semi-definite quadratic form, where matrices R and Q are semi definite positive and satify the following Liapunov's equation

$$A_0^T R A_0 - R = -Q.$$

Proof From theorem 2.1, it follows that system (2.1) is asymptotically y stable if and only if the μ type system in (2.3) is asymptotically stable. Therefore, for any given positive definite $(m+h) \times (m+h)$ matrix Q_1, there exists a positive definite $(m+h) \times (m+h)$ matrix R_1 such that $A_1^T R_1 A_1 - R_1 = -Q_1$.

Let $$R_2 = \begin{pmatrix} R_1 & \\ & 0 \end{pmatrix}\begin{matrix} m+h \\ p-h \end{matrix}, \quad Q_2 = \begin{pmatrix} Q_1 & \\ & 0 \end{pmatrix}\begin{matrix} m+h \\ p-h \end{matrix}$$

and $R = (L^{-1})^T R_2 L^{-1}$, $Q = (L^{-1})^T Q_2 L^{-1}$, then matrices R and Q are positive

Semi-definite, L is as above.

Let $V(x)=x^TRx$, $W(x)=-x^TQx$, then $\Delta V(x)|_{(2.1)}=W(x)$ and the following Liapunov's equation is satisfied.

$$A_0^TRA_0-R=-Q\ .$$

This completes the proof of the theorem.

§ 3 The Partial Stability

Consider the partial stability of the system

$$\begin{cases} y(\tau+1)=Ay(\tau)+Bz(\tau) \\ Z(\tau+1)=Cy(\tau)+Dz(\tau) \end{cases} \tag{3.1}$$

where $y=(x_1,\cdots,x_m)^T$, $z=(x_{m+1},\cdots,x_{m+p})^T$ $(p+m=n,\ p>0)$, A, B, C and D are the corresponding matrices.

Suppose that the μ-type of system (3.1) is described by (2.3).

Theorem 3.1. System (3.1) is y-stable if and only if μ-type system is stable, that is the norms of the characteristic roots of A_1 are no more than 1, if there is some root with norm 1, then the corresponding elementary divisor is of the order 1.

Proof We are only to prove the necessity. If there is some characteristic root λ of A_1 with its norm more than 1, then $\eta=(\eta_1^T,h_2^T)^T$, is the eigenvector of system (2.3) of λ, where h_1 is the eigenvector of $A_1\cdot L^{-1}\eta$ is the corresponding eigenvector of system (3.1) of λ, that means $\lambda^n L^{-1}\eta$ is a solution of system (3.1). According to the hypothesis of the theorem we can see that $\eta_1=\eta_2=\cdots=\eta_m=0$, that implies that $\eta_{m+1}=\cdots=\eta_{m+h}=0$. It is in controdiction with $h_1=(\eta_1,\cdots,\eta_{m+h})^T$ is a eigenvector of A_1 of λ.

In the same way as above, we can prove that if $|\lambda|=1$, then its elementary divisors are of order 1.

This completes the proof of the theorem.

Theorem 3.2. Linear discrete system (3.1) is stable and asymptotically y-stable if and only if there exists a positive definite quadratic form $V(x)=x^TRx$ such that $\Delta V(x)|_{(3.1)}=W(x)=-x^TQx$ is y-negative semi-definite quadratic form, where R and Q satisfy the following Liapunov's eqution

$$A_0^TRA_0-R=-Q\ .$$

Proof For convenient, we prove the theorem as the following cases.

Case 1. System (3.1) has a characteristic root $\lambda=1$ with the elementary divisor of order 1 (without loss of generality, we suppose that it has only one elementary divisor) and the rest of the roots are of norms less than 1.

In this case, we can reduce system (3.1) to (3.2) as follows by a nonsingular transformation.

$$\begin{cases} u(\tau+1)=A_2u(\tau) \\ x(\tau+1)=x(\tau) \end{cases} \tag{3.2}$$

where $u=(u_1,\cdots,u_{n-1})^T$, A_2 is a matrix of order $n-1$ and all the norms of the roots of det $(A_2-\lambda E)$ less than 1.

If we denote "T" as the transform matrix, then, for any positive definite matrix Q_1 of order $n-1$, there exists a positive definite matrix R_1 of order $n-1$ such that $A_2^TR_1A_2-R_1=-Q_1$.

Let $R_2=\mathrm{diag}(R_1,1)$, $Q_2=\mathrm{diag}(Q_1,0)$, and

$R=(T^{-1})^TR_2T^{-1}$, $Q=(T^{-1})^TQ_2T^{-1}$; then

$V(x)=x^TR\,x$ is a positive definite quadratic form such that $\Delta V(x)|_{(3.1)}=W(x)=-x^TQx$ is y-negative semi-definite quadratic form, R and Q satisfy the following Liapunov's equation.

$$A_0^TRA_0-R=-Q.$$

Case 2 System (3.1) has a single characteristic root $\lambda=-1$, the rest of the the roots are of norms less than 1.

Case 3 System (3.1) has a single pair of complex characteristic roots $a\pm bi$ satisfying $a^2+b^2=1$, the rest of the roots are of norms less than 1.

We can prove case 2 in the same way as case 1.

From the fact that $\Delta(x^2+y^2)\Big|_{\begin{cases} x(\tau+1)=ax(\tau)-by(\tau) \\ y(\tau+1)=bx(\tau)+ay(\tau)\end{cases}}=(ax(\tau)-by(\tau))^2+(ay(\tau)+bx(\tau))^2-(x^2(\tau)+y^2(\tau))=0$, we can show that the theorem holds for case 3.

In the same way as in cases 1,2 and 3, we can prove that the theorem holds for the rests of the cases.

This completes the proof of the theorem.

Theorem 3.3. System (3.1) is asymptotically $x_1,\cdots,x_m$-stable, and $x_{m+1},\cdots,x_{m+1}$-stable if and only if there exists a $x_1,\cdots,x_{m+1}$-positive semi-definite quadratic form $V(x)=x^TRx$ such that $\Delta V(x)|_{(3.1)}=W(x)=-x^TQx$ is $x_1,\cdots,x_m$-negative semi-definite quadrqtic form, R and Q satisfy the following Liapunov's equation.

$$A_0^TRA_0-R=-Q$$

Proof According to theorem 3.1, we can reduce system (3.1) to (3.3) as follows.

$$\begin{cases} u(\tau+1)=A_1u(\tau) \\ \theta(\tau+1)=C_1u(\tau)+D_1\theta(\tau) \end{cases} \tag{3.3}$$

where $u=(u_1,\cdots,u_{m+1},u_{m+1+1},\cdots,u_{m+l+h_1})^T$, $\theta=(\theta_1,\cdots,\theta_{p_1-h_1})^T$ $(m+l+p_1=n,\ p_1>0,\ h_1>0)$. A_1, C_1 and D_1 are the corresponding matrices, A_1 satisfies the conditions in theorem 3.3, then there exists a positive quadratic form $V_1(u)=u^TR_1u$ such that $\Delta V_1(u)|_{u(\tau+1)=A_1u(\tau)}=W_1(u)=-u^TQ_1u$ is $u_1,\cdots,u_m$-negative semi-definite, R_1

and Q_1 satisfy the following equation.

$$A_1^T R_1 A_1 - R_1 = -Q_1 .$$

Let $R_2 = \mathrm{diag}(R_1, 0)$, $Q_2 = \mathrm{diag}(Q_1, 0)$, $R = (L^{-1})^T R_2 L^{-1}$, $Q = (L^{-1})^T Q_2 L^{-1}$, where L is the transform matrix from (3.1) to (3.3), then $V(x) = x^T R x$ and $W(x) = -x^T Q x$ satisfy the claims of the theorem.

This completes the proof of the theorem.

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totically y-stable is equivalent to that of all variables if and only if that rank $K_p(t) = h = p$.

Corollary 2 Under the hypothesis of theorem 2, if system (1) has m negative characteristic exponents, then it is asymptotically y-stable if and only if $B(t) \equiv 0$ in system (1).

Remark The theorems in this paper generalize the results in PMM., 42 (1978), No.2., C. 269—271. (Russian).