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Weighted Weak Type Hardy's Inequalities

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§ Ⅰ. Introduction

Suppose $1\leq p,q<\infty$, $f(x,y)$, $U(x,y)$ and $V(x,y)$ are nonnegative measurable functions on $(0,\infty)\times(0,\infty)$. The Hardy operators P and Q are defined by

$$Pf(x,y)=\int_0^x\int_0^y f(t,s)\,ds\,dt,$$
$$Qf(x,y)=\int_x^\infty\int_y^\infty f(t,s)\,ds\,dt. \qquad (1,1)$$

In the definition given below, the operator T expresses P or Q.

Definition Let $1\leq p,q<\infty$, we say that $(U、V)$ is a strong type (p,q) weight pair for T, if there is a constant $C>0$ independent of f so that the following holds

$$\left[\int_0^\infty\int_0^\infty (Tf(x,y))^q U(x,y)\,dy\,dx\right]^{1/q}\leq C\left[\int_0^\infty\int_0^\infty f(x,y)^p V(x,y)\,dy\,dx\right]^{1/p}, \qquad (1.2)$$

and we say that $(U、V)$ is a weak type $(p、q)$ weight pair for T, if there is a constant $C>0$ independent of f so that for all $a>0$ and all $f>0$ the following holds

$$\left[\iint_{\{(x,y):Tf(x,y)>a\}} U(x,y)\,dy\,dx\right]^{1/q}\leq C/a\left[\int_0^\infty\int_0^\infty f(x,y)^p V(x,y)\,dy\,dx\right]^{1/p}. \qquad (1.3)$$

The smallest possible constant C in (1.2) and (1.3), is called the strong and weak norms of T, expressed as $\|T\|_S$ and $\|T\|_W$, respectively.

In 1978, B. Muckenhoupt pointed out [1] that it is an interesting and difficult open question to give the weight character of the higher dimensional weighted Hardy's inequalities. He pointed out that in two dimensions, if a weight pair $(U、V)$ for T is the strong type $(p、q)$ weight pair, then the following holds:

$$\sup_{\substack{r>0\\ s>0}}\left(\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\right)\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\,dy\,dx\right)^{p-1}<\infty, \qquad (1.4)$$

the converse, however, is false. There is an example of $(U、V)$ that satisfies (1.4), but (1.2) can't hold for all f.

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In 1984, B. Muckenhoupt raised this question again [2].

Although (1.4) isn't the weight character of the strong type weighted inequalities for the operator P, it will be proved in this paper that (1.4) is the weight character of the weak type weighted inequalities for the operator P.

Throughout the paper, $1/p+1/p'=1$ and $0.\infty$ is taken as 0.

Theorem 1 If $1<p, q<\infty$, then (U,V) is a weak type (p,q) weight pair for P, if and only if

$$K=\sup_{\substack{r>0\\ s>0}}\left(\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\right)^{1/q}\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\,dy\,dx\right)^{1/p'}<\infty. \quad (1.5)$$

Moreover, $K=\|P\|_w$.

Theorem 2 If $1<p, q<\infty$, then (U,V) is a weak type (p,q) weight pair for Q, if and only if

$$J=\sup_{\substack{r>0\\ s>0}}\left[\int_0^r\int_0^s U(x,y)\,dy\,dx\right]^{1/q}\left[\int_r^\infty\int_s^\infty V(x,y)^{-1/(p-1)}\,dy\,dx\right]^{1/p'}<\infty. \quad (1.6)$$

Moreover, $J=\|Q\|_W$.

Theorem 1 and theorem 2 given above are only the results when $n=2$, but these results can be extended to $n>2$ by the method given in this paper.

§ 2. Proof of Theorem 1

We begin with the necessity part. Let $r>0$, $s>0$ and denote $h(r,s)=\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\,dy\,dx\right)^{1/p'}$.

i) If $h(r,s)=0$, then $K=0$ by the convention.

ii) If $h(r,s)=\infty$, then $V(x,y)^{-1/p}$ isn't in $L^{p'}[(0,r)\times(0,s)]$. Hence there is a nonnegative $g(x,y)$ in $L^p[(0,r)\times(0,s)]$ with $g(x,y)V(x,y)^{-1/p}$ nonintegrable on $(0,r)\times(0,s)$. Now, if

$$f(x,y)=\begin{cases} g(x,y)V(x,y)^{-1/p} & \text{on } (0,r)\times(0,s)\\ 0 & \text{elsewhere}, \end{cases}$$

then $Pf(x,y)=\int_0^x\int_0^y f(\omega,\tau)\,d\tau\,d\omega=\int_0^r\int_0^s g(\omega,\tau)V(\omega,\tau)^{-1/p}\,d\tau\,d\omega=\infty$, for $x>r, y>s$.

Therefore $(r,\infty)\times(s,\infty)\subset\{(x,y):Pf(x,y)>a\}$ for any $a>0$. Since (U,V) is the weak type (p,q) weight pair for P and $a>0$ is arbitrary,

$$\left(\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\right)^{1/q}\leq\left[\iint_{\{(x,y):Pf(x,y)>a\}}U(x,y)\,dy\,dx\right]^{1/q}$$

$$\leq\|P\|_W/a\left(\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dy\,dx\right)^{1/p}$$

$$=\|P\|_W/a\left(\int_0^r\int_0^s g(x,y)^p\,dy\,dx\right)^{1/p}=\|P\|_W/a\,\|g\|_{L^p}=0.$$

Thus $K=0$ in this case.

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iii) Suppose then that $0<h(r,s)<\infty$. If $p=1$, let $\varepsilon>0$ and select a set E with positive measure $|E|$, $E\subset(0,r)\times(0,s)$, so that

$$(*)\qquad V(x,y)\leqslant\varepsilon+\operatorname*{essinf}_{(0,r)\times(0,s)}V(x,y)$$

for all $(x,y)\in E$.

If $f(x,y)$ is the characteristic function of E and $x\geqslant r, y\geqslant s$, then $Pf(x,y)=|E|$. When m is large enough we have $a=|E|-1/m>0$ and $[r,\infty)\times[s,\infty)\subset\{(x,y):P(x,y)>|E|-1/m\}$. Since (U,V) is the weak type $(1,q)$ weight pair for operator P, this implies

$$\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\leqslant\iint_{\{(x,y):Pf(x,y)>|E|-1/m\}}U(x,y)\,dy\,dx$$

$$\leqslant\|P\|_W(|E|-1/m)^{-q}\left(\int_0^\infty\int_0^\infty f(x,y)V(x,y)\,dy\,dx\right)^q.$$

Making $m\to\infty$ in this inequality and observing $(*)$, we have

$$\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\leqslant\|P\|_W|E|^{-q}\left(\iint_E V(x,y)\,dy\,dx\right)^q$$

$$\leqslant\|P\|_W|E|^{-q}\left[\varepsilon+\operatorname*{essinf}_{(0,r)\times(0,s)}V(x,y)\right]^q|E|^q,$$

and

$$\left(\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\right)^{1/q}\left(\operatorname*{essinf}_{(0,r)\times(0,s)}V(x,y)\right)^{-1}\leqslant\|P\|_W<\infty.$$

Hence we obtain $K\leqslant\|P\|_W$ when $p=1$.

If $p>1$, we denote

$$f(x,y)=\begin{cases}V(x,y)^{-1/(p-1)} & \text{on } (0,r)\times(0,s)\\ 0 & \text{elsewhere,}\end{cases}$$

then $Pf(x,y)=\int_0^x\int_0^y f(\omega,\tau)\,d\tau\,d\omega=\int_0^r\int_0^s V(\omega,\tau)^{-1/(p-1)}\,d\tau\,d\omega=[h(r,s)]^{p'}$ for $x\geqslant r$, $y\geqslant s$.

If m is large enough, then $a=[h(r,s)]^{p'}-1/m>0$ and $[r,\infty)\times[s,\infty)\subset\{(x,y)::Pf(x,y)>h(r,s)^{p'}-1/m\}$. Since (U,V) is the weak type (p,q) weight pair for operator P, this implies

$$\left[\int_r^\infty\int_s^\infty U(x;y)\,dy\,dx\right]^{1/q}\leqslant\left[\iint_{\{(x,y):Pf(x,y)>a\}}U(x,y)\,dy\,dx\right]^{1/q}$$

$$\leqslant\|P\|_W a^{-1}\left(\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dy\,dx\right)^{1/p}$$

$$=\|P\|_W[h(r,s)^{p'}-1/m]^{-1}\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\,dy\,dx\right)^{1/p}$$

Let $m\to\infty$ in this inequality, notice that $h(r,s)^{p'}=\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\,dy\,dx$,

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then we obtain

$$\left(\int_r^\infty\int_s^\infty U(x,y)\,dy\,dx\right)^{1/q}\leqslant\|P\|_W\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\,dy\,dx\right)^{-1/p'}.$$

Thus, in and case, we have $K\leqslant\|P\|_W<\infty$.

Next we prove the sufficiency part of theorem 1. If $\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dy\,dx=\infty$, then the result is obvious, hence we may spppose $\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dy\,dx<\infty$.

First we prove the result of the theorem when $\int_0^\varepsilon\int_0^\delta f(x,y)\,dy\,dx=\infty$ for an any $\varepsilon>0$, $\delta>0$.

In fact, we must have $\int_0^\varepsilon\int_0^\delta V(x,y)^{1-p'}dy\,dx=\infty$ for any $\varepsilon>0$, $\delta>0$ in this case.

If it is not so, because $\int_0^\varepsilon\int_0^\delta [V(x,y)^{-1/p}]^{p'}dy\,dx=\int_0^\varepsilon\int_0^\delta V(x,y)^{1-p'}dy\,dx<\infty$, then $V(x,y)^{-1/p}$ is in $L^{p'}((0,\varepsilon)\times(0,\delta))$. Hence

$$\int_0^\varepsilon\int_0^\delta [f(x,y)V(x,y)^{1/p}]^p dy\,dx\leqslant\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dy\,dx<\infty$$

and this yields $f(x,y)V(x,y)^{1/p}\in L^p((0,\varepsilon)\times(0,\delta))$. From Holder inequality we have $f(x,y)=[f(x,y)V(x,y)^{1/p}][V(x,y)^{-1/p}]\in L^1((0,\varepsilon)\times(0,\delta))$, but this is contradictory to $\int_0^\varepsilon\int_0^\delta f(x,y)\,dy\,dx=\infty$. Therefore we must have

$$\int_0^\varepsilon\int_0^\delta V(x,y)^{1-p'}dy\,dx=\infty \quad \text{for any } \varepsilon>0,\ \delta>0.$$

Since the weight pair (U,V) satisfies (1.5), we have $\int_\varepsilon^\infty\int_\delta^\infty U(x,y)\,dy\,dx=0$ for any $\varepsilon>0$, $\delta>0$ by the convention. This implies $U(x,y)=0$ almost everywhere on $(0,\infty)\times(0,\infty)$.

Thus, for any $a>0$, we obtain

$$\left[\iint_{\{(x,y):Pf(x,y)>a\}}U(x,y)\,dy\,dx\right]^{1/q}\leqslant C/a\left[\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dy\,dx\right]^{1/p},$$

therefore the conclusion holds in this case.

Next we will give the proof of the theorem when there are $r>0$, $s>0$ such that

$$0<\int_0^r\int_0^s f(x,y)\,dy\,dx=A<\infty.$$

Because $f(x,y)$ is integrable on $(0,r)\times(0,s)$ in this case, hence $Pf(x,y)$ is a continuous function on $(0,r)\times(0,s)$. Therefor, for any a, $0<a<A$, there must be a point (a,b) in $(0,r]\times(0,s]$ so that $Pf(a,b)=a$.

Now we construct a sequence of function.

$$g_n(x,y)=\begin{cases} f(x,y) & \text{on } (0,a]\times(0,b]\cup[a/2^n,\infty)\times[b,\infty)\cup[b/2^n,b]\times(a,\infty) \\ 0 & \text{elsewhere}\end{cases}\quad n=0,1,2,\cdots$$

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Obviously the conditions below are satisfied by sequence of function.

i) $0 \leqslant g_n(x,y) \leqslant g_{n+1}(x,y)$, $n=0,1,2,\cdots$ and $\lim\limits_{n\to\infty} g_n(x,y)=f(x,y)$ on $(0,\infty)\times(0,\infty)$;

ii) $Pg_n(x,y) \leqslant Pg_{n+1}(x,y)$, $n=0,1,2,\cdots$ and $\lim\limits_{n\to\infty} Pg_n(x,y)=Pf(x,y)$ on $(0,\infty)\times(0,\infty)$;

iii) $\{(x,y): Pg_n(x,y)>a\} \subset \{(x,y): Pg_{n+1}(x,y)>a\} \subset \{(x,y): Pf(x,y)>a\}$, $n=0,1,2,\cdots$ on $(0,\infty)\times(0,\infty)$;

iv) for any n and $(x,y)\in(0,a]\times(0,b]$, $Pg_n(x,y) \leqslant Pg_n(a,b)=Pf(a,b)=a$, in particular, $Pg_n(a/2^n, b/2^n) \leqslant Pg_n(a,b)=a$.

From this we deduce $\{(x,y): Pg_n(x,y)>a\} \subset [a/2^n,\infty)\times[b/2^n,\infty)$ and

$$\iint\limits_{\{(x,y):Pg_n(x,y)>a\}} U(x,y)\,dydx \leqslant \int_{a/2^n}^{\infty}\int_{b/2^n}^{\infty} U(x,y)\,dydx$$

$$\leqslant K^q\left[\int_0^{a/2^n}\int_0^{b/2^n} V(x,y)^{1-p'}dydx\right]^{-q/p'}$$

$$\leqslant (K/a)^q[Pg_n(a/2^n,b/2^n)]^q\left[\int_0^{a/2^n}\int_0^{b/2^n} V(x,y)^{1-p'}dydx\right]^{-q/p'},$$

by Holder inequality yields

$$[Pg_n(a/2^n,b/2^n)]^q=\left(\int_0^{a/2^n}\int_0^{b/2^n} g_n(x,y)\,dydx\right)^q$$

$$\leqslant\left[\int_0^{a/2^n}\int_0^{b/2^n} g_n(x,y)^pV(x,y)\,dydx\right]^{q/p}\left[\int_0^{b/2^n}\int_0^{a/2^n}\int_0^{b/2^n} V(x,y)^{1-p'}dydx\right]^{q/p'}.$$

Thus,

$$\left[\iint\limits_{\{(x,y):Pg_n(x,y)>a} U(x,y)\,dydx\right]^{1/q} \leqslant K/a\left[\int_0^{\infty}\int_0^{\infty} f(x,y)^pV(x,y)\,dydx\right]^{1/p}.$$

Making $n\to\infty$, we obtain

$$\left[\iint\limits_{\{(x,y):Pf(x,y)>a\}} U(x,y)\,dydx\right]^{1/q} \leqslant K/a\left[\int_0^{\infty}\int_0^{\infty} f(x,y)^pV(x,y)\,dydx\right]^{1/p}$$

for $0<a<A$.

If $a \geqslant A$, then we may construct a sequence of function:

$$h_n(x,y)=\begin{cases} f(x,y) & \text{on } (0,r]\times(0,s]\cup[r/2^n,\infty)\times(s,\infty)\cup[s/2^n,s]\times(r,\infty) \\ 0 & \text{elsewhere}\end{cases}$$

$$n=0,1,2,\cdots$$

and the sequence of function is provided with the same properties of $\{g_n(x,y)\}$.

By the method given in the proof for $0<a<A$, we still have in $a \geqslant A$

$$\left[\iint\limits_{\{(x,y):Ph_n(x,y)>a\}} U(x,y)\,dydx\right]^{1/q} \leqslant K/a\left(\int_0^{\infty}\int_0^{\infty} f(x,y)^pV(x,y)\,dydx\right)^{1/p}$$

therefore,

$$\left[\iint\limits_{\{(x,y):Pf(x,y)>a\}} U(x,y)\,dydx\right]^{1/q} \leqslant K/a\left(\int_0^{\infty}\int_0^{\infty} f(x,y)^pV(x,y)\,dydx\right)^{1/p},$$

as $n\to\infty$.

Summing up the discussion above we know that if the weight pair (U,V) satisfies (1.5), then for any $a>0$ and $f(x,y)$ we must have

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$$\left[\iint_{\{(x,y):Pf(x,y)>a\}} U(x,y)\,dydx\right]^{1/q} \leqslant K/a\left(\int_0^\infty\int_0^\infty f(x,y)\ V(x,y)\,dydx\right)^{1/p}.$$

Since $K<\infty$, hence (U, V) is the weak type (p,q) weight pair for operator p and $K \geqslant \|P\|_W$. This is the proof of the sufficiency part.

From the process of the proving of the necessity and sufficiency we know $K=\|P\|$. The proof of theorem 1 is completed.

§ 3. Proof of Theorem 2

First let us give a lemma.

Lemma $(U(x,y), V(x,y))$ is a strong (or weak) type (p,q) weight pair for operator Q if and only if $(1/(xy)^2U(1/x,1/y),\ (x,y)^{2(p-1)}V(1/x,1/y))$ is the strong (or weak) type (p,q) weight pair for operator P.

Proof By the definition of operators P and Q we have

$$Qf(x,y)=\int_x^\infty\int_y^\infty f(t,s)\,dsdt$$
$$=\int_0^{1/x}\int_0^{1/y} f(1/t,1/s)1/(st)^2\,dsdt=Pg(1/x,1/y),$$

where $g(x,y)=1/(xy)^2f(1/x,1/y)$, therefore

$$\left[\int_0^\infty\int_0^\infty(Qf(x,y))^qU(x,y)\,dydx\right]^{1/q}=\left[\int_0^\infty\int_0^\infty(Pg(1/x,1/y))^qU(x,y)\,dydx\right]^{1/q}$$
$$=\left[\int_0^\infty\int_0^\infty(Pg(x,y))^q1/(xy)^2U(1/x,1/y)\,dydx\right]^{1/q}$$

and for any $a>0$

$$\iint_{\{(x,y):Qf(x,y)>a\}} U(x,y)\,dydx=\iint_{\{(1/x,1/y):Pg(1/x,1/y)>a\}} U(x,y)\,dydx$$
$$=\iint_{\{(x,y):Pg(x,y)>a\}} 1/(xy)U(1/x,1/y)\,dydx$$

and

$$\left(\int_0^\infty\int_0^\infty f(x,y)^pV(x,y)\,dydx\right)^{1/p}$$
$$=\left(\int_0^\infty\int_0^\infty[f(1/x,1/y)]^pV(1/x,1/y)1/(xy)^2\,dydx\right)^{1/p}$$
$$=\left(\int_0^\infty\int_0^\infty[1/(xy)^2f(1/x,1/y)]^p(xy)^{2(p-1)}V(1/x,1/y)\,dydx\right)^{1/p}$$
$$=\left(\int_0^\infty\int_0^\infty g(x,y)^p(xy)^{2(p-1)}\ V(1/x,1/y)\,dydx\right)^{1/p}.$$

From the relationship above we can deduce the result of lemma immediately.

The conclusion of theorem 2 can be obtained from theorem 1 and lemma. The proof of theorem 2 is omitted.

References

[1] B. Mukenhoupt, Proc.Symp. in pure Math., 35(1) (1979), 69-83.

[2] B. Mukenhoupt, "Weighted norm inequalities", Lecture notes in Math., 1043, 318-321.

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加权弱型Hardy不等式

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摘　　要

设$1\leqslant p,q<\infty$，$f(x,y)$，$U(x,y)$，$V(x,y)$是$(0,\infty)\times(0,\infty)$上的非负可测函数。记

$$(1.1)\qquad Pf(x,y)=\int_0^y\int_0^x f(t,s)\,\mathrm{d}s\mathrm{d}t$$
$$Qf(x,y)=\int_x^\infty\int_y^\infty f(t,s)\,\mathrm{d}s\mathrm{d}t$$

称算子P、Q为二维Hardy 算子。在下面的定义中，算子T表P或Q。

定义　如存在常数$C>0$，使对一切$f(x,y)$成立着

$$(1.2)\qquad \left(\int_0^\infty\int_0^\infty [Tf(x,y)]^q U(x,y)\mathrm{d}y\mathrm{d}x\right)^{1/q}\leqslant C\left(\int_0^\infty\int_0^\infty f(x,y)^p V(x,y)\mathrm{d}y\mathrm{d}x\right)^{1/p}$$

则称(U,V)关于算子T是强(p,q)型权对；如存在常数$C>0$，使对一切$f(x,y)$和$a>0$，成立着

$$(1.3)\qquad \left[\iint_{\{(x,y):Tf(x,y)>a\}} U(x,y)\mathrm{d}y\mathrm{d}x\right]^{1/q}\leqslant Ca^{-1}\left(\int_0^\infty\int_0^\infty f(x,y)^p V(x,y)\mathrm{d}y\mathrm{d}x\right)^{1/p}$$

则称(U,V)关于算子T是弱(p,q)型权对。使（1.2）、（1.3）式成立的最小常数C，分别称为T的强范数和弱范数，记作$\|T\|_S$、$\|T\|_W$。

1978年，B. Muckenhoupt 指出[1]，给出高维形式的加权Hardy不等式的权函数特征，是个有意义而又困难的问题。他指出，仅在二维的情形下，如权对(U,V)关于算子P是强(p,p)型权对，则有下述结果：

$$(1.4)\qquad \sup_{\substack{r>0\\ s>0}}\left(\int_r^\infty\int_s^\infty U(x,y)\mathrm{d}y\mathrm{d}x\right)\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\mathrm{d}y\mathrm{d}x\right)^{p-1}<\infty。$$

但有反例表明，当$1<p<\infty$时，存在满足(1.4)式的(U,V)，却不能使(1.2)式对一切f成立。

1984年，B. Muckenhoupt 再次提出上述问题[2]。

虽（1.4）式不再是算子P的强型加权不等式的权函数特征，然而本文证明了，它却完全刻划了P的弱型加权不等式的权对特征。

定理1　$1\leqslant p,q<\infty$，则(U,V)关于算子P为弱(p,q)型权对的充分必要条件为

$$(1.5)\qquad K\equiv\sup_{\substack{r>0\\ s>0}}\left(\int_r^\infty\int_s^\infty U(x,y)\mathrm{d}y\mathrm{d}x\right)^{1/q}\left(\int_0^r\int_0^s V(x,y)^{-1/(p-1)}\mathrm{d}y\mathrm{d}x\right)^{1/p'}<\infty,$$

此外　　$K=\|P\|_W$。

定理2　$1\leqslant p,q<\infty$，则(U,V)关于算子Q为弱(p,q)型权对的充分必要条件为

$$(1.6)\qquad J\equiv\sup_{\substack{r>0\\ s>0}}\left(\int_0^r\int_0^s U(x,y)\mathrm{d}y\mathrm{d}x\right)^{1/q}\left(\int_r^\infty\int_s^\infty V(x,y)^{-1/(p-1)}\mathrm{d}y\mathrm{d}x\right)^{1/p'}<\infty,$$

此外，　　$J=\|Q\|_W$。

上述定理1、定理2仅是$n=2$的结果，但用文中的方法完全可将其推广至$n>2$的情形。

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