Journal of Mathematical Resecrh and Exposition
Vol. 10 No.1 Feb. 1990

Primary Modules Determined by Their Lattice of Submodules*

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When is a module determined by its lattice of submodules? Baer gave two elegant results as follows.

Theorem A[1]. Let G be a p-group with final rank $\geqslant 3$ or a bounded p-group containing three independent elements of maximal order. Then every isomorphism of the lattice of subgroups $L(G)$ onto $L(H)$ is induced by an isomorphism from G onto H, where H is a p-group and G and H are commutative.

Theorem B[2]. Let $V_i, i=1,2$, be a finite dimensional vector space over a division ring Δ_i and assume the lattice of subspaces $L(V_1)\simeq L(V_2)$ and $\dim(V_1)\geqslant 3$.Then $\Delta_1\simeq\Delta_2$ and $\dim(V_1)=\dim(V_2)$. Moreover, any isomorphism of $L(V_1)$ onto $L(V_2)$ is induced by a bijective semi-linear map of V_1 onto V_2.

In fact, there exists an unified form of the above two results as follows.

Assume $R_i, i=1,2$, is a MLPI ring (i.e., R_i is an associative ring with unit 1, and its every maximal left ideal is a principal ideal), p_i is an elment of R_i such that $R_i p_i$ is a maximal left ideal of R_i, M_i is a p_i-primary R_i-module [3], simply, primary module. Then we have

Theorem C. Assume the lattice of submodules $L(M_1)\overset{f}{\simeq}L(M_2)$ and final Goldie dimension $M_1\geqslant 3$. Then there exists an isomorphism $\{\psi_n: R_1/R_1p_1^n\to R_2/R_2p_2^n(n\in\boldsymbol{N})\}$ between inverse system[4] $\{R_1/R_1p_1^n(n\in\boldsymbol{N});\ \theta_n\}$ and inverse system $\{R_2/R_2p_2^n(n\in\boldsymbol{N});\ \theta_n'\}$, where θ_n and θ_n' $(n\in\boldsymbol{N})$ are canonical epimorphic, $\psi_n(n\in\boldsymbol{N})$ is an isomorphism of rings. Let R_1^* and R_2^* be the inverse limit of these inverse systems, respectively. Then there exists ring isomorphism ψ from R_1^* onto R_2^* and a bijective ψ-linear map of M_1 onto M_2 which is inducing f, where final Goldie dimension $M_1=\min\limits_{n=0,1,2,\cdots}$ Goldie dimension $p_1^n M_1$.

When M_1 is bounded with three independent elements of maximal order, we obtain a result which is similar to Theorem C

Throughout this paper we discuss always under the hypotheses of Theorem C.

We begin the proof of Theorem C with the following.

* Received Mar. 25, 1988.

Proposition 1[5] Let R_1 be a ring, M_1 be a primary R_1-module. Then for each $0 \neq a \in M_1$ there exists a natural number $n \in \boldsymbol{N}$ such that

$$\text{and}_{R_1}(a) = R_1 p_1^n \neq R_1 p_1^{n-1} .$$

In this case, n is called the order of a and denoted by $o(a)$. Define $o(0)=0$. It is clear that M_1 is bounded if and only if the set $\{o(a) | a \in M_1\}$ is finite.

Proposition 2. Let R_1 and R_2 be rings, $M_1 = R_1 a_1$ be a primary R_1-module. If $o(a_1)=n$, then $L(M_1) \simeq [0, n-1]$. Furthermore, if M_2 is a primary R_2-module such that $L(M_2) \simeq L(M_1)$, then $M_2 = R_2 a_2$ and $o(a_2) = n$.

Proof Let A be any non-trivial submodule of M_1, then there is an element $0 \neq b = r p_1^t a_1 \in A$ with $r \in R_1 - R_1 p_1$ and $0 < t < n$. By the definition of $R_1 p_1$, $R_1 = R_1 r + R_1 p_1 = \cdots = R_1 r + R_1 p_1^n$, and hence $1 = ur + vp^n (u, v \in R_1)$. Thus $p_1^t a_1 = ub \in A$. Let k be the least natural number such that $p_1^k a_1 \in A$, then $A = R_1 p_1^k a_1$ since for each $c \in A$ with $c = r p_1^t a_1$ and $r \in R_1 - R_1 p_1$, we get $p_1^t a_1 \in A$, and hence $t \geqslant k$ and $c \in R_1 p_1^k a_1$. Therefore, $A = R_1 p_1^k a_1$ and $L(M_1) \simeq [0, n-1]$.

Suppose $L(M_1) \simeq L(M_2)$, then $L(M_2)$ is a chain, and hence for each $b \in M_2$ with $o(b)=t$, $L(R_2 b) \simeq [0, t-1]$ is a sublattice of $[0, n-1]$. Thus $t \leqslant n$. Let $a_2 \in M_2$ such that $o(a_2) = m = \max\{o(b) | b \in M_2\}$. We claim that $M_2 = R_2 a_2$ and $m = n$.

Since $L(M_2)$ is a chain, for each $b \in M_2$ with $o(b) = t \leqslant m$, we have (1) $R_2 b \leqslant R_2 p_2^{m-t} a_2$, or 2) $R_2 b > R_2 p_2^{m-t} a_2$. We only prove that (2) can not hold. Otherwise, $p_2^{m-t} a_2 = rb$ for some $r \in R_2$. If $r \in R_2 - R_2 p_2$, then there exists an element $u \in R_2$ such that $b = urb = u p_2^{m-t} a_2 \in R_2 p_2^{m-t} a_2$, a contradiction. On the other hand, if $r \in R_2 p_2$, i.e., $r = r_2 p_2$ for some $r_2 \in R_2$, then $o(rb) = o(r_2 p_2 b) \leqslant t-1 < t = o(p_2^{m-t} a_2) = o(rb)$, a contradiction, too, Hence $M_2 = R_2 a_2$, and $m = n$. since .

$$[0, m-1] \simeq L(M_2) \simeq L(M_1) \simeq [0, n-1]. \qquad \blacksquare$$

Lemma 3 Let M_1 be a primary R_1-module such that $M_1 = X \oplus Y$ where X and Y are cyclic primary modules, and let f be an isomorphism of $L(M_1)$ onto $L(M_2)$. To any generators x', y' of $f(X)$, $f(Y)$ one can find genrators x, y of X, Y such that $f R_1(x+y) = R_2(x'+y')$. Here $x+y$ may be any generator of $f^{-1} R_2(x'+y')$.

Lemma 4 Let $M_1 = R_1 u \oplus R_1 x$ such that $o(x) \leqslant o(u)$. If $L(M_1) \overset{f}{\simeq} L(M_2)$ with $f R_1 u = R_2 u'$, then there exists one and only one $x' \in M_2$ such that

$$f R_1 x = R_2 x' \text{ and } f R_1(x+u) = R_2(x'+u').$$

Conveniently, we may write $x' = \varphi(x; u, u', f)$.

Lemma 5 Assume $M_1 = R_1 u \oplus A$ such that $\operatorname{ann}_{R_1}(A) < R_1 p_1^{o(u)}$. If $L(M_1) \overset{f}{\simeq} L(M_2)$ and $f R_1 u = R_2 u'$, then the mapping

$$x \mapsto x' = \varphi(x; u, u', f) \ (x \in A)$$

is one-to-one between A and $f(A)$.

Sometimes, we denote $\varphi(x; u, u', f)$ by $\varphi(x)$.

Lemma 6 Under the hypotheses of Lemma 5, if $x, y \in A$ and u, x, y are indepent (i.e., $R_1u+R_1x+R_1y=R_1u\oplus R_1x\oplus R_1y$), then

$$\varphi(x+y;u,u',f)=\varphi(x;u,u',f)+\varphi(y;u,u',f). \qquad (*)$$

The proofs of the above Lemmas 3-6 refer to [1].

Lemma 7 Assume in addition to the hypotheses of Lemma 6 that to each $z\in A$ there exists a $w(\in A)$ independent of u,z with $o(z)\leqslant o(w)$. Then $(*)$ holds for every pair $x,y\in A$.

Proof First we prove $\varphi(rz)=\varphi(z)+\varphi(r-1)z$ for all $z\in A$ and all $r\in R_1$. Choose a w as stated. Then $z, w-z$ and u, further $w+z$, $(r-1)z$ and u are independent, so that Lemma 6 implies $\varphi(w)=\varphi[(w-z)+z]=\varphi(w-z)+\varphi(z)=\varphi(w)+\varphi(-z)+\varphi(z)$ and hence $\varphi(-z)=-\varphi(z)$, and $\varphi(w)+\varphi(rz)=\varphi(w+rz)=\varphi[(w+z)+(r-1)z]=\varphi(w+z)+\varphi[(r-1)z]=\varphi(w)+\varphi(z)+\varphi[(r-1)z]$ implies $\varphi(rz)=\varphi(z)+\varphi[(r-1)z]$.

In particular, $r=n\cdot 1$ $(n\in\mathbf{Z})$, by induction, we obtain $\varphi(nz)=n\varphi(z)$.

Now if x,y are arbitrary elements of A, not loss generality, we may assume $o(y)\leqslant o(x)$. By the following Proposition 8, there exists an $r\in R_1$ such that $y_1=y+rx$ and x are independent, and hence $\varphi(y)=\varphi(y_1-rx)=\varphi(y_1)-\varphi(rx)$, $\varphi(x)=\varphi(rx)-\varphi[(r-1)x]=\varphi(rx)+\varphi[(1-r)x]$, $\varphi(x+y)=\varphi[(y+rx)+(1-r)x]=\varphi(y_1)+\varphi[(1-r)x]=\varphi(x)+\varphi(y)$.

Proposition 8 Let $M_1=R_1x+R_1y$ with $k:=o(y)\leqslant o(x)=:s$. Then there exists an $r\in R_1$ such that $M_1=R_1x\oplus R_1y_1$ where $y_1=y+rx$.

Proof Since $R_1p_1^{k-1}y$ and $R_1p_1^{s-1}x$ are irreducible, we have the following:

(i) If $R_1p_1^{k-1}y\neq R_1p_1^{s-1}x$, then $M_1=R_1x\oplus R_1y$;

(ii) If $R_1p_1^{k-1}y=R_1p_1^{s-1}x$, then $p_1^{k-1}y=u_1p_1^{s-1}x=p_1^{s-1}u_1'x$ (since $R_1p_1=p_1R_1$) for some $u_1,u_1'\in R_1$. Let $y_1=y-p_1^{s-k}u_1'x$, then $M_1=R_1x+R_1y_1$ but $k_1=o(y_1)\leqslant k-1<o(y)$. Assume $R_1x+R_1y_1$ is not direct sum, then $R_1p_1^{k-1}y_1=R_1p_1^{s-1}x$, similarly, there exists a $y_2=y_1-u_2x$ $(u_2\in R_1)$ such that $M_1=R_1x+R_1y_2$ but $k_2=o(y_2)<o(y_1)$.

It is clear that there is a $y_t=y_{t-1}-u_tx=y+rx(u_t,\ r\in R_1)$ such that $M_1=R_1x\oplus R_1y_t$. ■

Lemma 9 Let M_1 be a primary R_1-module such that $\mathrm{ann}_{R_1}(M_1)=R_1p_1^k$, and suppose u,v,w are independent elements of order k in M_1. If $L(M_1)\overset{f}{\simeq}L(M_2)$ and $fR_1u=R_2u'$, then the following three statements hold:

(i) The mapping $x\to\varphi(x;u,u',f)$ induces a unique group isomorphism φ: $M_1\simeq M_2$ such that $\varphi(u)=u'$.

(ii) There exist ring isomorphisms

$$\psi_t: R_1/R_1p_1^t\to R_2/R_2p_2^t,\quad t=1,2,\cdots,k$$

and commutative diagrams:

$$\begin{array}{ccc} R_1/R_1p_1^t & \xrightarrow{\psi_t} & R_2/R_2p_2^t \\ \theta_t\downarrow & & \downarrow\theta_t' \\ R_1/R_1p_1^{t-1} & \xrightarrow{\psi_{t-1}} & R_2/R_2p_2^{t-1} \end{array} \qquad (1)$$

where θ_t and θ_t' $(t=1,2,\cdots,k)$ are canonical homomorphisms.

(iii) $\varphi(rx)=\varphi(\overline{rx})=\psi_t(\overline{r})\cdot\varphi(x)$, where $t=o(x)$, $\overline{r}$ is the image of r under the natural mapping $R_1\longrightarrow R_1/R_1p_1^t$.

Proof (i) Firstly, by $_{R_1}M_1=R_1p_1^k$, there exist elements $z_a\in M_1$ such that $M_1=R_1u\oplus R_1v\oplus R_1w\oplus(\oplus_a R_1z_a)$. By Lemma5, the mapping $x\to\varphi(x;u,u',f)=:\varphi(x)$ is a group isomorphism from $A(=R_1v+R_1w+(\sum_a R_1z_a))$ onto $f(A)$ and $M_2=R_2u'\oplus f(A)$. For every $r\in R_1$, $\varphi(rv)=r'\varphi(v)$ with $r'+R_2p_2^k$ is unique in $R_2/R_2p_2^k$ since $R_1rv\leqslant R_1v$ and fR_1v is cyclic R_2-module. We may assume that $0'=0$, $1'=1$.

Next for each $z\in R_1w+(\sum_a R_1z_a)$. Denote $\varphi(rz)=r''\varphi(z)$ with $r''+R_2p_2^t$ is unique in $R_2/R_2p_2^t$, where $t=o(z)$. We also assume $0''=0$ and $1''=1$.Then

$$r'\varphi(z)=r''\varphi(z)\quad(r\in R_1),\tag{2}$$

since $\varphi(rv)-\varphi(rz)=\varphi[r(v-z)]$, we have $r'\varphi(v)-r''\varphi(z)=r^*[\varphi(v)-\varphi(z)]$ for some $r^*\in R_2$, and $r'-r^*\in R_2p_2^k$ and $r''-r^*\in R_2p_2^t$. It follows that $r'-r''\in R_2p_2^t$ and $r'\varphi(z)=r''\varphi(z)$. Therefore $\varphi(r_1u+r_1v+r_1w+\sum_a r_az_a)=r_1'\varphi(u)+r_2'\varphi(v)+r_3'\varphi(w)+\sum_a r_a'\varphi(z_a)$.

It is clear that φ is a group isomorphism of M_1 onto M_2.

(ii) Now we prove $R_1/R_1p_1^k\simeq R_2/R_2p_2^k$. For every pair $a,b\in R_1$, $R_2(u'+(a+b)'v'+w')\leqslant R_2(u'+a'v')+R_2(b'v'+w')$ since

$$R_1(u+(a+b)v+w)\leqslant R_1(u+av)+R_1(bv+w).$$

But the only element of the form $u'+cv'+w'$ contained in the righthand side is $u'+(a'+b')v'+w'$. It follows that

$$(a+b)'v'=(a'+b')v',\quad (a+b)'-(a'+b')\in R_2p_2^k.$$

Similarly, using the fact that $R_1(u+abv+w)\leqslant R_1u+R_1(bv+w)$, we can conclude that $(ab)'v'=a'b'v'$, $(ab)'-(a'b')\in R_2p_2^k$. Define $\psi_k:R_1/R_1p_1^k\to R_2/R_2p_2^k$, given by $\psi_k(a+R_1p_1^k)=a'+R_2p_2^k$, we can easily check that ψ_k is a ring homomorphism. We claim that ψ_k is isomorphic. ψ_k is monic, since for each $a\notin R_1p_1^k$, $R_1(v+aw)\neq R_1v$, R_1w. It follows $R_2(v'+a'w')\neq R_2v'$, R_2w' and $a'\notin R_2p_2^k$. ψ_k is epic, since f^{-1}: $L(M_2)\simeq L(M_1)$ and $f^{-1}R_2u'=R_1u$, $f^{-1}R_2v'=R_1v$, $f^{-1}R_2(u'+v')=R_1(u+v)$, for each $a\in R_2$, we obtain $f^{-1}R_2av'=R_1av$, $f^{-1}R_2(u'+av')=R_1(u+av)$, hence $R_2av'=fR_1av=R_2a'v'$, and $R_2(u'+av')=fR_1(u+av)=R_2(u'+a'v')$, $r^*(u'+av')=u'+a'v'$ for some $r^*\in R_2$. Thus $r^*-1\in R_2p_2^k$, $av'=a'v'$ and $a-a'\in R_2p_2^k$. Therefore ψ_k is isomorphic.

On the other hand, if $fR_1z=R_2z'$, by Proposition 2, then $o(z)=o(z')$. Suppose $z=p_1^{k+t}v$, then $o(z)=t=o(z')$. If $\varphi(rz)=r''\varphi(z)$ $(r\in R_1)$, $0''=0$ and $1''=1$, by (2), $r'\varphi(z)=r''\varphi(z)$. Similarly, we obtain a ring isomorphism $\psi_t:R_1/R_1p_1^t\to R_2/R_2p_2^t$, given by $\psi_t(a+R_1p_1^t)=a''+R_2p_2^t$ when we consider u, $y(=p_1^{k-t}w)$ and z instead of u,v and w, and a commutative diagram

$$\begin{array}{ccc} R_1/R_1p_1^t & \xrightarrow{\psi_t} & R_2/R_2p_2^t \\ \downarrow & & \downarrow \\ R_1/R_1p_1^k & \xrightarrow{\psi_k} & R_2/R_2p_2^k \end{array}$$

where the column homomorphisms are natural homomorphisms. Moreover, we have commutative diagrams

$$\begin{CD} R_1/R_1p_1^t @>{\psi_t}>> R_2/R_2p_2^t \\ @V{\theta_t}VV @VV{\theta'_t}V \\ R_1/R_1p_1^{t-1} @>{\psi_{t-1}}>> R_2/R_2p_2^{t-1} \end{CD}$$

$t=1,2,\cdots,k$.

(iii) By (ii), we obtain immediately $\varphi(\overline{rx})=\varphi(rx)=r'\varphi(x)=\psi_t(\bar{r})\cdot\varphi(x)$, where $t=o(x)$, $\bar{r}$ is the image of r under the canonical mapping $R_1\to R_1/R_1p_1^t$.

Finally, the above $\varphi(\varphi(u)=u')$ is unique, this proof refers to [1]. ■

Now we give the proof of Theorem C.

For a fixed $k\in\boldsymbol{N}$, consider $M_1(p_1^k)$ and $M_2(p_2^k)$. It is clear that f induces an isomorphism: $L(M_1[p_1^k])\simeq L(M_2[p_2^k])$. By Lemma 9, there is a group isomorphism φ_k of $M_1[p_1^k]$ onto $M_2[p_2^k]$ and there are ring isomorphisms ψ_t of $R_1/R_1p_1^t$ onto $R_2/R_2p_2^t(t=1,2,\cdots,k)$, which satisfy (1). Choose elements $w,v\in M_1$ with $o(v)=k+1=o(w)$. It follows that $u=p_1v\in M_1[p_1^k]$. Denote $\varphi_k(p_1v)=u'$. Let $fR_1w=R_2w''$, then, by Lemma 9, there is a unique group isomorphism $\varphi'_{k+1}(\varphi'_{k+1}(w)=w'')$ from $M_1[p_1^{k+1}]$ into $M_2[p_2^{k+1}]$. By the definition of φ'_{k+1}, we have

$$fR_1v=R_2v'' \text{ and } fR_1(w+v)=R_2(w''+v''),$$
$$fR_1p_1v=R_2p_1''v'' \text{ and } fR_1(w+p_1v)=R_2(w''+p_1''v'').$$

On the other hand, $\varphi_k(p_1v)=u'$, i.e., $fR_1p_1v=R_2u''$. It follows that $R_2u'=R_2p_1''v''$ and $u'=r_2p_1''v''$ for some $r_2\in R_2-R_2p_2$. Thus there exists an $r_1\in R_2$ such that

$$r_1r_2-1\in R_2p_2^k \tag{3}$$

We need another isomorphism φ_{k+1} from $M_1[p_1^{k+1}]$ into $M_2[p_2^{k+1}]$ such that $\varphi_k(p_1v)=\varphi_{k+1}(p_1v)$. By (3), the following hold.

$$R_2w''=R_2r_2w'',\ R_2v''=R_2r_2v'',\ R_2(w''+v'')=R_2(r_2w''+r_2v''),$$
$$R_2p_1''v''=R_2r_2p_1''r_1\cdot r_2v'' \text{ and } R_2(w''+p_1''v'')=R_2(r_2w''+r_2p_1''r_1\cdot r_2v'').$$

By Lemma 9, there is a unique isomorphism φ_{k+1} from $M_1[p_1^{k+1}]$ into $M_2[p_2^{k+1}]$ such that $\varphi_{k+1}(v)=r_2v''$ and $\varphi_{k+1}(p_1v)=r_2p_1''r_1$. $\varphi_{k+1}(v)=r_2p_1''r_1\cdot r_2v''=r_2p_1''v''=u'=\varphi_k(p_1v)$. Thus $\varphi_{k+1|M_1[p_1^k]}=\varphi_k$.

Remark. (1) If $\varphi'_{k+1}(rv)=r''\varphi'_{k+1}(v)=r''v''(r\in R_1)$, then

$$\varphi_{k+1}(rv)=r_2r''r_1\varphi_{k+1}(v).$$

(2) φ_{k+1} is inducing a ring isomorphism

$$\psi_{k+1}:R_1/R_1p_1^{k+1}\to R_2/R_2p_2^{k+1}$$

given by $\psi_{k+1}(r+R_1p_1^{k+1})=r_2r''r_1+R_2p_2^{k+1}$, and ψ_{k+1} and $\psi_1,\cdots,\psi_k$ satisfy (1).

Thus there are two classes of isomorphisms:

$$\varphi_1,\cdots,\ \varphi_k,\cdots;\ \psi_1,\cdots,\ \psi_k,\cdots$$

such that $\varphi_{k+1|M_1[p_1^k]}=\varphi_k$ $(k=1,2,\cdots)$ and $\psi_1,\cdots,\psi_k,\cdots$ satisfy (1). Therefore there exists a mapping φ agreeing with φ_k on $M_1[p_1^k]$ for every k. Since $M_1 = \bigcup_{k=1}^{\infty} M_1[p_1^k]$, we get an isomorphism of M_1 onto M_2, and φ and $\psi_1, \cdots, \psi_k, \cdots$ satisfy the conditions of Lemma 9. Since the following diagram commutative

$$\begin{array}{ccccccccc}
\cdots \rightarrow & R_1/R_1p_1^{k+1} & \xrightarrow{\theta_k} & R_1/R_1p_1^k & \rightarrow \cdots & \xrightarrow{\theta_1} & R_1/R_1p_1 & \rightarrow 0 \\
 & \downarrow{\scriptstyle \psi_{k+1}} & & \downarrow{\scriptstyle \psi_k} & & & \downarrow{\scriptstyle \psi_1} & \\
\cdots \rightarrow & R_2/R_2p_2^{k+1} & \xrightarrow{\theta_k'} & R_2/R_2p_2^k & \rightarrow \cdots & \xrightarrow{\theta_1'} & R_2/R_2p_2 & \rightarrow 0
\end{array}$$

we get a ring isomorphism ψ of R_1^* onto R_2^*, given by

$$\psi : r^* = (r_k + R_1p_1^k) \rightarrow (\psi_k(r_k) + R_2p_2^k),$$

where $R_i^* = \lim_k (R_i/R_ip_i^k, \theta_k)$ is the inverse limit of the inverse system $\{R_i/R_ip_i^k, \theta_k\}$, $i = 1, 2$.

On the other hand, M_i can be regarded as R_i^*-module in a natural way[3], $i = 1, 2$. and the following holds.

$$\varphi(r^*x) = \psi(r^*)\varphi(x) \quad \text{for all } r^* \in R_1^*, \text{ all } x \in M_1.$$

Thus φ is indeed a ψ-linear isomorphism of M_1 onto M_2 and is inducing the lattice isomorphism $f : fR_1x = R_2\varphi(x)$. ■

Conclusion If $p_1 = 0 = p_2$, then Theorem C implies Theorem B; if $R_1 = \mathbf{Z} = R_2$ and p_1 and p_2 are two prime numbers, then Theorem C implies that $p_1 = p_2$ and Theorem A.

References

[1] Fuschs, L., Abelian Groups. Publ. House of the Acad. Sci. Budapest. 1958.

[2] Jacobson, N., Basic Algebra I. W. H. Freeman and Company. 1980.

[3] Kaplansky, I., Infinite Abelian Group. Ann Arbor. 1954.

[4] Fan Yun, Modules over Group Algebras II (Representation and Homology). Lecture for graduate students. Wuhan University. 1986.

[5] Zhou Borong, Modules Determined by Their Socles. Preprint

由子模格决定的准素模

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摘要 本文主要给出以下定理C．

设R_i $(i=1,2)$是MLPI环（即R_i是有位单元的结合环，且每个极大左理想必是主理想），元素$p_i\in R_i$使得R_ip_i是R_i的极大左理想，M_i是p_i-准素的R_i-模[3]．则我们有以下

定理C 设M_1的终Goldie维数（$=\min\{p_1^nM_1$的Goldie维数$\mid n=0,1,2,\cdots\}$）$\leqslant 3$.如果有子模格同构$f:L(M_1)\simeq L(M_2)$．则有逆向全射系[4] $\{R_1/R_1p_1^n\ (n\in \mathbf{N});\theta_n\}$与$\{R_2/R_2p_2^n\ (n\in \mathbf{N});\theta'_n\}$之间的同构$\{\psi_n:R_1/R_1p_1^n\to R_2/R_2p_2^n\ (n\in \mathbf{N})$，其中$\theta_n$和$\theta'_n\ (n\in \mathbf{N})$是自然满同态，$\psi_n(n\in \mathbf{N})$是环同构．若令$R_1^*,R_2^*$分别是以上两逆向全射系的逆向极限环．则有环同构$\psi:R_1^*\simeq R_2^*$ 和M_1到M_2的ψ-线性同构φ，φ诱导出$f:fR_1x=R_2\varphi(x)$，$\forall x\in M_1$．

易见：

（1）当$p_1=0=p_2$，且M_1是有限维向量空间时，由定理C即得射影几何的基本定理[2]；

（2）当$R_1=\mathbf{Z}=R_2$，且p_1和p_2为素数时，由定理C即得$p_1=p_2$，从而得Baer关于交换p-群的相应结果[1]．