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Estimation of Eigenvalues of Product of two Self-Conjugate Semi-Pósitive Definite Quaternions Matrices*

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In this paper we gave a definition of norm of quaternion matrix, at the base of this we estimated eigenvalues of product of two self-conjugate semi-positive definite quaternions matrices, generalized and improved correcponding results for hermitian positive definite matrices in [5].

Let Q denote real quaternion field. If $x=a+bi+cj+dk\in Q$, where a, b, c, d are real numbers, we write $\overline{X}=a-bi-cj-dk$ and $N(X)=a^2+b^2+c^2+d^2$. Let Q denote the set of all $m\times n$ matrices, $GL_n(Q)$ denote genenal linear group over Q, $SC_n(Q)$ denote the set of all $n\times n$ self-conjugate matrices. If $P\in Q^{n\times n}$, then P^* denote trasposed conjugate matrix of P. We also write $A\geqslant 0$ ($A>0$), if A be a semi-positive definite (positive definite) self-conjugate matrix.

Lemma 1 Suppose $B=\begin{pmatrix}B_1 & B_2^*\\ B_2 & B_3\end{pmatrix}\geqslant 0$ then equation $B_1X^*=B_2^*$ have a solution for X.

Prove since $B\geqslant 0$, hence $B_1\geqslant 0$ by proposition 1 in [1]. Moreover, by [2] we known there exists a generalized unitary matrix U such that $UB_1U^*=\operatorname{diag}(\lambda_1,\cdots,\lambda_t)\geqslant 0$. Now it is easy to see that

$$\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}B_1 & B_2\\ B_2 & B_3\end{pmatrix}\begin{pmatrix}U^* & 0\\ 0 & I\end{pmatrix}=\begin{pmatrix}UB_1U^* & UB_2^*\\ B_2U^* & B_3\end{pmatrix}\geqslant 0.$$

Using Lemma 4 in [4], we can knew, if $\lambda_i=0$ then all elements of the ith-rows of $(UB_1U^*\ \ UB_2^*)$ are zero. It is easy to see that

$$\begin{aligned}\operatorname{rank}(B_1\ \ B_2^*)&=\operatorname{rank}\ U(B_1\ \ B_2^*)\begin{pmatrix}U^* & 0\\ 0 & I\end{pmatrix}\\ &=\operatorname{rank}(UB_1U^*\ \ UB_2^*)=\operatorname{rank}B_1\end{aligned}$$

Evidantly this implies that equation $B_1X^*=B_2^*$ have a solution by [3].

Definition 1 Suppose λ_1 is maximum eigenvalue of A^*A, where $A\in Q^{m\times n}$, then $\|A\|=\lambda_1^{1/2}$ is said the norm of the matrix A.

Definition 2 If $A\in Q^{n\times n}$, $X\in Q^{n\times 1}$, $\lambda\in Q$, $AX=\lambda X$ then λ is called eigenvalue

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of A, and X is called correspunding eigenvector of A.

Lemma 2 If $A\in Q^{m\times n}$, $x\in Q^{n\times 1}$, then $\|A\|=\sup\limits_{\|x\|=1}\|Ax\|$.

Prove Since $A^*A\geq 0$, it is easy to see that all eigenvalues $\lambda_1, \cdots, \lambda_n$ of A^*A, are nonnegative. Suppose $\lambda_1\geq\cdots\geq\lambda_n\geq 0$ and corresponding eigenvectors $X_1, \cdots, X_n$ are a generalized orthonormal system. $X\in Q^{n\times 1}$, we write $X=\sum\limits_{i=1}^{n}X_i a_i$, where $a_i\in Q$. Obviously, if $\|X\|=1$ then we have $\sum\limits_{i=1}^{n}N(a_i)=1$. Moreover,

$$\|AX\|^2=(AX)^*AX=X^*A^*AX=(\sum_{i=1}^{n}\overline{a_i}X_i^*)A^*A(\sum_{i=1}^{n}X_i a_i)$$

$$=(\sum_{i=1}^{n}\overline{a_i}X_i^*)(\sum_{i=1}^{n}\lambda_i X_i a_i)=\sum_{i=1}^{n}\lambda_i N(a_i)\leq\lambda_1\sum_{i=1}^{n}N(a_1)=\lambda_1$$

and if $X=X_1$ then we have $X^*A^*AX=X_1^*A^*AX_1=\lambda_1X_1^*X_1=\lambda_1$, hence

$$\sup_{\|X\|=1}\|AX\|=\lambda_1^{1/2}=\|A\|.$$

Lemma 3 If $A\in Q^{m\times n}$, $B\in Q^{n\times p}$, then $\|AB\|\leq\|A\|\|B\|$.

Prove We can write $\|aA\|=a^2\|A\|$ for arbitrary real number a. Suppose $\|BX\|=a\neq 0$ $\forall$ $X\in Q^{p\times 1}$ and $\|X\|=1$, then it is clear $\|a^{-1/2}BX\|=1$, hence

$$\|ABX\|=\|A(a^{-1/2}BX)\|a\leq a\sup_{\|y\|=1}\|Ay\|=a\|A\|=\|BX\|\|A\|\leq\|B\|\|A\|.$$

Moreover, inequality $\|AB\|\leq\|A\|\|B\|$ is proved.

Theorem Suppose $0\neq A\in SC_n(Q)$, $0\neq B\in SC_n(Q)$; λ_i and μ_i are, respectively, eigenvalues of A and B, write $|\lambda_1|\geq\cdots\geq|\lambda_n|$ and $|\mu_1|\geq\cdots\geq|\mu_n|$; λ be arbitrary eigenvalues of AB, then λ be real number and we have the following

(i) If $A\geq 0$, $A\notin GL_n(Q)$ and $B\geq 0$, $B\notin GL_n(Q)$ then $\lambda\leq\lambda_1\mu_1$.

(ii) If $A>0$ or $B>0$ then $|\lambda|\leq|\lambda_1\mu_1|$, in particular if $A>0$ and $B>0$ then $\lambda\leq\lambda_1\mu_1$

(iii) If $A>0$ and $B\in GL_n(Q)$, or $B>0$ and $A\in GL_n(Q)$ then $|\lambda|\geq|\lambda_n||\mu_n|$ in particular if $A>0$ and $B>0$ then $\lambda\geq\lambda_n\mu_n$.

Prove (i) Suppose rank $A=r<n$ and rank $B<n$, since $A\geq 0$ hence we know from [2], there is a generalized unitary matrix U such that $UAU^*=\begin{pmatrix}D&0\\0&0\end{pmatrix}$, where $D=\text{diag}(d_1, \cdots, d_r)$, $d_i>0$ $\forall i=1,\cdots, r$.

Since $\begin{pmatrix}D^{-1/2}&0\\0&I\end{pmatrix}\begin{pmatrix}D&0\\0&0\end{pmatrix}\begin{pmatrix}D^{-1/2}&0\\0&I\end{pmatrix}=\begin{pmatrix}I_r&0\\0&0\end{pmatrix}$. Hence we have

$$\begin{pmatrix}D^{-1/2}&0\\0&I\end{pmatrix}UAU^*\begin{pmatrix}D^{-1/2}&0\\0&I\end{pmatrix}=\begin{pmatrix}I_r&0\\0&0\end{pmatrix}, \qquad (1)$$

where $D^{-1/2}=\text{diag}(d_1^{-1/2}, \cdots, d_r^{-1/2})$. Since $B\geq 0$ we also have

$$\begin{pmatrix} D^{1/2} & 0 \\ 0 & I \end{pmatrix} UBU^* \begin{pmatrix} D^{1/2} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} B_1 & B_2^* \\ B_2 & B_3 \end{pmatrix} \geqslant 0 \tag{2}$$

where $B_1 \in SC_r(Q)$ and $B_1 > 0$, write $L = \begin{pmatrix} I_r & 0 \\ -X_0 & I \end{pmatrix}$, where X_0 satisfies the equation $B_1 X^* = B_2^*$ (Lemma 1). By derict calculations, It is easy to see that

$$L\begin{pmatrix} D^{1/2} & 0 \\ 0 & I \end{pmatrix} UBU^* \begin{pmatrix} D^{1/2} & 0 \\ 0 & I \end{pmatrix} L^* = \begin{pmatrix} B_1 & 0 \\ 0 & B_3 - X_0 B_2^* \end{pmatrix} \tag{3}$$

and

$$L^{*-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

We write $U_0 B_1 U_0^* = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$, $\Sigma = \mathrm{diag}(b_1, \cdots, b_t, 0, \cdots, 0)$, where $t \leqslant r$; $b_i > 0$ $(i = 1, \cdots, t)$ are eigenvalues of B_1; U_0 is a generalized unitary matrix. From formula (3) and (4), we have

$$\begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D^{1/2} & 0 \\ 0 & I \end{pmatrix} UBU^* \begin{pmatrix} D^{1/2} & 0 \\ 0 & I \end{pmatrix} L^* \begin{pmatrix} U_0^* & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & * \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix} L^{*-1} \begin{pmatrix} D^{-1/2} & 0 \\ 0 & I \end{pmatrix} \cdot UAU^* \begin{pmatrix} D^{-1/2} & 0 \\ 0 & I \end{pmatrix} L^{-1} \begin{pmatrix} U_0^* & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

Both sides of (6) left multiplied by both side of (5), we obtained the result: AB similar to $\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$. This showed non-zero eigenvalues of AB are same as non-zero eigenvalues of B_1, hence λ be real number. Suppose $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, where $U_1 \in Q^{n\times n}$, by (2), it foilows that

$$\begin{pmatrix} D^{-1/2} B_1 D^{-1/2} & * \\ * & * \end{pmatrix} = UBU^* = \begin{pmatrix} U_1 B U_1^* & * \\ * & * \end{pmatrix}$$

Moreover, $B_1 = D^{1/2} U_1 B U_1^* D^{1/2}$. By Lemma 3, we have

$$\lambda \leqslant \|B_1\| \leqslant \|D^{1/2}\| \|U_1\| \|B\| \|U_1^*\| \|D^{1/2}\| \leqslant \|D\| \|B\| = \lambda_1 \mu_1$$

(ii) It suffices to show that if $A > 0$ then $|\lambda| \leqslant |\lambda_1 \mu_1|$. We assme $A = U^* D U$, where D be positive diagonal matrix, U is a generalized unitary matrix by $A > 0$. It is clear $D^{-1/2} UAU^* D^{-1/2} = I_n$. At the same time we have $D^{1/2} UBU^* D^{1/2} \geqslant 0$. Moreover, there exists a generalizsd unitary matrix U_1 such that

$$U_1^* D^{1/2} UBU^* D^{1/2} U_1 = \mathrm{diag}(\lambda_1', \cdots, \lambda_n') \tag{7}$$

and

$$U_1^* D^{-1/2} UAU^* D^{-1/2} U_1 = I_n. \tag{8}$$

Both sides of (8) left multiplied by both side of (7), it follows that $\lambda_1', \cdots, \lambda_n'$ are eigenvalues of AB, hence

$$|\lambda| \leqslant \max_i \{|\lambda_i'|\} = \|U_1^* D^{1/2} UBU^* D^{1/2} U_1\| \leqslant \|D\| \|B\| = |\lambda_1| |\mu_1| = \lambda_1 |\mu_1|$$

The second statement is clear.

(iii) In this case, since $(AB)^{-1}=B^{-1}A^{-1}$, hence $|\lambda^{-1}|\leqslant|\lambda_n^{-1}||\mu_n^{-1}|$ by the results of (ii), it follows that $|\lambda|\geqslant|\lambda_n||\mu_n|$. Last statement is clear.

Remark 1 When A and B are hermition positive definite matrices, the result in [5] is $\frac{2}{n}(\lambda_n^2+\mu_n^2)^{-1}\lambda_n^2\mu_n^2\leqslant\lambda<\frac{n}{2}(\lambda_1^2+\mu_1^2)$, but our theorem is better than that in [5]. For example suppose $n=20$, $\lambda_1=1$, $\mu_1=10$, $\lambda_n=0.1$, $\mu_n=1$ we ha have $\frac{0.001}{1.01}<\lambda<1010$ by [5], however we have $0.1<\lambda<10$ by our theorem.

Remark 2 If $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&3\end{pmatrix}$, $B=\begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix}$ then $AB=\begin{pmatrix}1&0&0\\1&0&0\\0&0&3\end{pmatrix}$. By direct calculation it is easy to see that minimal positive eigenvalue of A, B, AB, are 2,1,1 and obviously $1\ngeqslant 1\times 2$. This fact shown in case (i) of theorem we cannot derive $\lambda\geqslant\lambda_0\mu_0$, where λ_0 and μ_0 are minimal positive eigenvalue of A and B.

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两个四元数自共轭半正定矩阵乘积的特征估计

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摘要 设A和B均非0的n阶实四元数自共轭矩阵，λ_i及μ_i分别为共特征值（$i=1,\cdots,n$），且规定$|\lambda_1|\geqslant|\lambda_2|\geqslant\cdots\geqslant|\lambda_n|$，$|\mu_1|\geqslant|\mu_2|\geqslant\cdots\geqslant|\mu_n|$，又$\lambda$为$AB$之任意特征值，则$\lambda$为实数，且（1）若$A\geqslant 0$，$A\notin GL_n(Q)$，$B\geqslant 0$，$B\notin GL_n(Q)$，则$\lambda\leqslant\lambda_1\mu_1$；（2）若$A>0$或$B>0$，则$|\lambda|\leqslant|\lambda_1\mu_1|$，特别当$A>0$且$B>0$时有$\lambda\leqslant\lambda_1\mu_1$；（3）若$A>0$，$B\in GL_n(Q)$，或$B>0$，$A\in GL_n(Q)$则$|\lambda|\geqslant|\lambda_n\mu_n|$，特别当$A>0$且$B>0$时有$\lambda\geqslant\lambda_n\mu_n$.