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A Note on the Connectedness of the Hyperspace with Hausdorff Metric*

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Abstract In this note, some examples and propositions are given to answer the open question posed by E. Klein & A. C. Thompson in [1]. The main results in this note is that: if X is a metric space and there exists an arc in X, then

(1) there exists an order arc a in $P(X)$ such that $a(0)$ is connected and $\bigcup_{0<t<1} a(t)$ is not connected;

(2) there exists a point $x \in X$ and an arc β in $F(X-\{x\})$ such that $\beta(0) \in C(X-\{x\})$ and $\bigcup \beta \notin C(X-\{x\})$.

Let X be a topology space, we let

$$P(X)=\{A\subset X;\ A\neq\varnothing\};\quad F(X)=\{F\in P(X);\ F \text{ is closed}\};$$
$$C(X)=\{E\in F(X);\ E \text{ is connected}\}.$$

The Vietoris topology on $P(X)$ $(F(X), C(X))$ is denoted by T_v. If X is a metrie space, the Hausdorff metric topology on $P(X)$ is denoted by T_h. In this note, when we write $P(X)$ we mean the space $P(X)$ with the Hausdorff metric topology, and $F(X)$ $(C(X))$ means the space $F(X)$ $(C(X))$ with the relative topology of $P(X)$.

The hyperspaces with the Vietoris topology are generally well bahaved with respect to connectivity. For example, the following statement is true.

Theorem A If a is a connected subset of $(P(X), T_v)$ and if some element of a is connected, then $\bigcup a$ is connected ([1]).

If X is a compact metric space, then the Vietoris topology and the Hausdorff metric topology coincide on $F(X)$ [1]. The following result is well known.

Theorem B If X is a continuum, a is an order arc in $F(X)$ beginning with $A\in C(X)$, then $a\subset C(X)$ and $\bigcup a\in C(X)$ [2].

In [1], E. Klein and A. C. Thompson posed the following question

Question A Whether or not the analogue of Theorem A is valid for the Hausdorff topology on $F(X)$ (or $P(X)$) (See [1] pp. 48—52).

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The following question is similar to Question A.

Question B. Whether or not the analogue of Theorem **B** is valid for the space X which is not compact.

In this note we contruct two examples to establish that, in general, Theorem A is not valid for the Hausdorff metric topology and Theorem **B** is not valid for noncompact space.

By an arc we will mean a homemorphism a of a nondegenerate closed interval $[a,b]$, or we will mean the range $a([a,b])$ of such a homemorphism. An order arc in $F(X)$ ($P(X)$) is an arc a in $F(X)$ ($P(X)$) such that if $A, B\in a$, then $A\subset B$ or $B\subset A$.

Example 1 Let $X=D^2=\{(x,y); x^2+y^2\leqslant 1\}$, and let d be the usual metric on X. X is a continuum. Now, consider the following family of subsets in X: $A_0=\{(x,0): -1\leqslant x<0\}$; $A_t=\{(x,0); x\in[-1,0)\cup(0,t]\}$ $(0<t\leqslant 1)$. For $t\in[0,1]$, let $a(t)=A_t$, then a is a function from I in to $P(X)$. It is clear that $A_{t_1}\subset A_{t_2}$ if $t_1<t_2$. Denote $B(A,r)=\{E\in P(X); \delta(E,A)<r\}$, where δ is the Hausdorff metric indussed on $P(X)$ by d. One can readily check that for $s>0$, $a^{-1}(B(A_t,r))=I\cap(t-r, t+r)$, so a is an order are in $P(X)$. Then the family $\{A_t; 0\leqslant t\leqslant 1\}$ is a connected subset of $P(X)$, and A_0 is connected, but $\cup\{A_t: t\in[0,1]\}=[-1,0)\cup(0,1]$ is not connected.

The following statement is a direct consequence of above discussion,

Proposition 1 If X is a metric space and there exists an arc in X, then there exists an order arc a in $P(X)$ such that

(i) $a(0)$ is connected;

(ii) for $t>0$, $a(t)$ is not connected;

(iii) $\cup a=\cup a([0,1])$ is not connected.

Proof Let h be an are in X, without loss of generality we assume that the domain of h is $[-1,1]$. Let $F_0=[-1,0)$, $F_t=[-1,0)\cup(0,t]$. It is readily seen that the family $\{h(F_t); t\in[0,1]\}$ is a connected subset (order are) of $P(X)$. We define $a(t)=h(F_t)$, then (i)—(iii) follow from the fact that h is an injection.

A slight change in the above example provides the following example.

Example 2 Let $Y=D^2-\{(0,0)\}$, the family $\{A_t; t\in[0,1]\}$ and map a are same as which given in Example 1. It is clear that $A_t\in F(Y)$ and $a: I\to F(Y)$ is an order are beginning with $a(0)\in C(Y)$.

The preceding examples and Proposition 1 give

Proposition 2 If X is a metric space and there exists an arc in X, then there exists a subspace Y of X and an order arc a in $F(Y)$ such that

(i) $a(0)\in C(Y)$; (ii) for $t>0$, $a(t)\notin C(Y)$; (iii) $U_t a(t)\notin C(Y)$.

References

[1] E. Klein & a.C. Thompson, Theory of Correspondences, John Wiley & Sons, Inc., 1984.
[2] S.B. Nadler, Jr., Hyperspaces of Sets, Marcel Dekker, Inc., 1978.

关于超空间连通性的一个注记

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摘要 本文回答了由E. Klein与A. C. Thompson在其著作《Thery of Correspondences》中提出的一个问题．主要结果是：若X是一度量空间，且在X中存在一条弧，则在$P(X)$中有一条序弧α，满足条件 i) $\alpha(0)$是连通子集；ii) $\bigcup_{t}\alpha(t)$是不连通的．