Journal of Mathematical Research and Exposition
Vol.10 No.1 Feb. 1990

# Two Examples on Continuity of Multi-Value Mappings*

*Yu Jian*

(Guizhou Institute of Technology, Guiyang)

**Difinition 1**[1] Suppose $X$ and $Y$ are two Hausdorff topological spaces, $F:X\to 2^Y$, $\forall x\in X$, $F(x)\neq\varnothing$, $x_0\in X$, we say $F$ is upper semicontinuous ($u.s.c$) at $x_0$, if for any open set $G$ of $Y$ and $G\supset F(x_0)$, there is a neighbourhood $N(x_0)$ of $x_0$ such that $G\supset F(x)$ for all $x\in N(x_0)$. If $\forall x_0\in X$, $F$ is $u.s.c$ at $x_0$, we say $F$ is $u.s.c$ on $X$.

**Difinition 2**[2] Suppose $X$ is a Hausdorff topological space, $Y$ is a Hausdorff topological vector space, $F:X\to 2^Y$, $\forall x\in X$, $F(x)\neq\varnothing$. $x_0\in X$, we say $F$ is upper demicontinuous ($u.d.c$) at $x_0$, if for any open half-space $H$ of $Y$ and $H\supset F(x_0)$, there is a neighbourhood $N(x_0)$ of $x_0$ such that $H\supset F(x)$ for all $x\in N(x_0)$. If $\forall x_0\in X$, $F$ is $u.d.c$ at $x_0$, we say $F$ is $u.d.c$ on $X$.

**Difinition 3**[3] Suppose $X$ is a Hausdorff topological space, $Y$ is a Hausdorff topological vector space, $F:X\to 2^Y$, $\forall x\in X$, $F(x)\neq\varnothing$. $x_0\in X$, we say $F$ is upper hemicontinuous ($u.h.c$) at $x_0$, if for any $p\in Y^*$ (the vector space of all continuous linear functionals on $Y$), the function $\sigma(F(x),p)=\sup\limits_{y\in F(x_0)}(p,y)$ is upper semicontinuous at $x_0$. If $\forall x_0\in X$, $F$ is $u.h.c$ at $x_0$, we say $F$ is $u.h.c$ on $X$.

**Proposition** Suppose $X$ is a Hausdorff topological space, $Y$ is a Hausdorff topological vector space, $F:X\to 2^Y$, $\forall x\in X$, $F(x)\neq\varnothing$, then

(1) $F$ $u.s.c$ on $X\Rightarrow F$ $u.d.c$ on $X$.

(2) $F$ $u.d.c$ on $X\Rightarrow F$ $u.h.c$ on $X$.

**proof** (1) Since $H$ is open in $Y$, the conclusion is obvious.

(2) It is enough to prove $\forall x_0\in X$, $\forall x_n\in X$, $n=1,2,3,\cdots$, $x_n\to x_0$, $\forall p\in Y^*$, then

$$\sup_{y\in F(x_0)}\langle p,y\rangle\geqslant\varlimsup_{n\to\infty}\sup_{y\in F(x_n)}\langle p,y\rangle .$$

Without loss of generality, we can suppose $A=\sup\limits_{y\in F(x_0)}\langle p,y\rangle<+\infty$.

Suppose that the conclusion is false, then $\exists\,\varepsilon_0>0$, $\exists\, x_n\in X$, $n=1,2,3,\cdots$, $x_n\to x_0$ such that

$$A+\varepsilon_0<\lim_{n\to\infty}\sup_{y\in F(x_n)}\langle p,y\rangle .$$

* Received Jan. 19, 1988.

$\exists N, \forall n \geqslant N$, $A+\varepsilon_0 < \sup\limits_{y \in F(x_n)} \langle p, y\rangle$ and $\exists\, y_n \in F(x_n)$ such that $A+\varepsilon_0 < \langle p, y_n\rangle$,

Let $H=\{y \in Y: \langle p, y\rangle < A+\varepsilon_0\}$, it is an open half-space of $Y$ and $H \supset F(x_0)$, but $y_n \bar{\in} H$, so $H \not\supset F(x_n)$, $F$ isn't $u.d.c$ at $x_0$, a contradiction.

$F$ $u.d.c \not\Rightarrow F$ $u.s.c$, a example is given in a Hilbert space[4], but it is too complex.

In this paper, we given two examples in $R^2$, they are very simple: the first implies: $F$ $u.d.c \not\Rightarrow F$ $u.s.c$; the second implies $F$ $u.h.c \not\Rightarrow F$ $u.d.c$.

**Example 1** $X=\{(t,0)^T \in R^2: \frac{1}{2} \geqslant t \geqslant 0\}$, $\forall (t,0)^T \in X$, define $F\ (t,0)^T=\{(x_1,x_2)^T \in R^2: x_2 \geqslant (1-t)x_1^2\}$.

Suppose $H=\{(x_1,x_2)^T \in R^2: x_2 > ax_1-\delta\}$ is any open half-plane and $H \supset F(0,0)^T = \{(x_1, x_2)^T \in R^2: x_2 \geqslant x_1^2\}$, then $\delta > 0$ and since the line $x_2=ax_1-\delta$ does't meet the curve $x_2=x_1^2$, so $4\delta - a^2 > 0$. $\forall (t,0)^T \in X$, when $t < \frac{4\delta - a^2}{4\delta}$, the line $x_2=ax_1-\delta$, doesn't meet the curve $x_2=(1-t)x_1^2$, so $H \supset F(t,0)^T$, $F$ is $u.d.c$ at $(0,0)^T$.

Let $G=\{(x_1,x_2)^T \in R^2: x_2 > x_1^2-1\}$, it is an open set in $R^2$ and $G \supset F(0,0)^T$. $\forall (t, 0)^T \in X$ $(t \neq 0)$, when $x_1 > \sqrt{\frac{1}{t}}$, $x_1^2-1 > (1-t)x_1^2$, so $G \not\supset F(t,0)^T$, $F$ isn't $u.s.c$ at $(0, 0)^T$.

**Example 2** $X=\{(t,0)^T \in R^2: \frac{\pi}{4} \geqslant t \geqslant 0\}$, $\forall (t,0)^T \in X$, define $F(t,0)^T=\{(x_1, x_2)^T \in R^2: \frac{\pi}{2}+t > x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_2 \geqslant \operatorname{tg}(x_1-t)\}$. $\forall (t,0)^T \in X$, $\forall p=(p_1, p_2) \neq (0,0)$.

$$\sup_{(x_1,x_2)^T \in F(t,0)} (p_1x_1+p_2x_2) = \begin{cases} +\infty, & p_2>0. \\ p_1(\frac{\pi}{2}+t), & p_2=0, p_1>0. \\ 0, & p_2=0,\ p_1<0. \\ 0, & p_2<0,\ p_1 \leqslant 0. \\ p_1 t, & p_2<0, p_1>0, p_1 \leqslant |p_2| \quad (*) \\ p_1 t + p_1 \arccos\sqrt{-\frac{p_2}{p_1}} + p_2\sqrt{-\frac{p_1}{p_2}-1}, & p_2<0, \\ \quad p_1>0,\ p_1>|p_2| & (**) \end{cases}$$

The computations of $(*)$ and $(**)$:

$$\sup_{(x_1,x_2)^T \in F(t,0)^T} (p_1x_1+p_2x_2) = \max\Big(\sup_{(x_1,x_2)^T \in A_1} (p_1x_1+p_2x_2),\ \sup_{(x_1,x_2)^T \in A_2} (p_1x_1+p_2x_2)\Big)$$

where $A_1=\{(x_1, x_2)^T \in R^2: t \geqslant x_1 \geqslant 0,\ x_2 \geqslant 0\}$ $A_2=\{(x_1, x_2)^T \in R^2: \frac{\pi}{2}+t > x_1 \geqslant t,\ x_2 \geqslant \operatorname{tg}(x_1-t)\}$.

If $p_2<0,\ p_1>0$, $\sup\limits_{(x_1,x_2)^T \in A_1} (p_1x_1+p_2x_2)=p_1 t$. If $(x_1, x_2)^T \in A_2$, $p_1x_1+p_2x_2 \leqslant p_1x_1+p_2 \cdot \operatorname{tg}(x_1-t)=p_1(y+t)+p_2\operatorname{tg} y=p_1t+p_1y+p_2\operatorname{tg} y$, where $x_1-t=y$, $\frac{\pi}{2}>y \geqslant 0$.

(1) If $p_1 \leqslant |p_2|$, then $p_1t+p_1y+p_2\operatorname{tg} y \leqslant p_1t+p_1y+p_2y \leqslant p_1 t$. Since $(t,0)^T \in A_2$, $\sup\limits_{(x_1,x_2)^T \in A_2} (p_1x_1+p_2x_2)=p_1 t$ and therefore $\sup\limits_{(x_1,x_2)^T \in F(t,0)^T} (p_1x_1+p_2x_2)=p_1 t$.

(2) If $p_1>|p_2|$, since $\max\limits_{0<y<\frac{\pi}{2}}(p_1y+p_2\mathrm{tg}\,y)=p_1\mathrm{acrcos}\sqrt{-\frac{p_2}{p_1}}+p_2\sqrt{-\frac{p_1}{p_2}-1}$, so

$\sup\limits_{(x_1,x_2)^T\in A_2}(p_1x_1+p_2x_2)=p_1t+p_1\arccos\sqrt{-\frac{p_2}{p_1}}+p_2\sqrt{-\frac{p_1}{p_2}-1}$; since $(t,0)^T\in A_1\cap A_2$, so $p_1t+p_1\arccos\sqrt{-\frac{p_2}{p_1}}+p_2\sqrt{-\frac{p_1}{p_2}-1}\geqslant p_1t$ and therefore $\sup\limits_{(x_1,x_2)^T\in F(t,0)^T}(p_1x_1+p_2x_2)=$ $+p_1\arccos\sqrt{-\frac{p_2}{p_1}}+p_2\sqrt{-\frac{p_1}{p_2}-1}$. Thus since $\sup\limits_{(x_1,x_2)^T\in F(t,0)^T}(p_1x_1+p_2x_2)$ is continuous at $(0,0)^T$, so $F$ is $u.h.c$ at $(0,0)^T$.

Let $H=\{(x_1,x_2)^T\in\mathrm{R}^2:x_1<\frac{\pi}{2}\}$, it is an open half-plane and $H\supset F(0,0)^T$, $\forall(t,0)^T\in X$ $(t\neq 0)$, $H\not\supset F(t,0)^T$, so $F$ isn't $u.d.c$ at $(0,0)^T$.

## References


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# 关于多值映象连续性的两个例子


俞　建

（贵州工学院，贵阳）



**摘要**　本文在$\mathrm{R}^2$中构造了两个十分简单的多值映象的例子：第一个说明准上半连续$\not\Rightarrow$上半连续；第二个说明弱上半连续$\not\Rightarrow$准上半连续．