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Uniqueness of the Solution to the Operator Equation $f(A)=A$ *

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Abstract Let H be a complex Hilbert space, and let $f(z)=\sum_{n=0}^{\infty} B_n z^n$, $z\in\Delta=\{z: |z|<1\}$, where $\{B_n\}$ is a sequence of normal operators on H commmuting pairwise, such that $\|f(z)\|<1$ for $z\in\Delta$ and $1\overline{\in}\sigma(B_1)$. If $\exists T\in X=\{A\in \mathscr{B}(H)$: A commutes with every B_n and $\sigma(A)\subset\Delta\}$ with $f(T)=T$, then T is the unique one in X satisfying the equation, and T must be normal.

1. Notations

In general, we follow [1] or [2] for notation. For the convience of the reader, we recall that, as usual, $\mathscr{B}(H)$ denotes the C^*-algebra consisting of all bounded linear operators on a complex Hilbert space H. $\mathscr{N}_H(\Delta)$ will indicate the set of analytic operator-valued functions f from the open unit disc $\Delta=\{z\in\mathbf{C}: |z|<1\}$ into $\mathscr{B}(H)$ such that f takes on the form

$$f(z)=\sum_{n=0}^{\infty} B_n z^n \text{ for } z\in\Delta$$

where $\{B_n\}_{n=0}^{\infty}$ is a sequence of normal operators commuting pairwise and the series converges in the norm topology. For $T\in\mathscr{B}(H)$, T commuting with f is meant $Tf(z)=f(z)T$ for all z in Δ. If $\sigma(T)$ (the spectrum of T) is contained in Δ, $f(T)$ is defined by $f(T)=\sum_{n=0}^{\infty} B_n T^n$.

In this note, we obtain an improvement of the relative theorem in [2] on the uniqueness of the solution to the operator equation $f(A)=A$.

2. Main result

Theorem. Let H be a complex Hilbert space and $f\in\mathscr{N}_H(\Delta)$. Suppose f satisfies the following conditions:

(i). $\|f(z)\|<1$ for $z\in\Delta=\{z: |z|<1\}$; (ii). $1\overline{\in}\sigma(f'(0))$;

(iii). There exists a T in $X=\{A\in\mathscr{B}(H)$: A commutes with f and $\sigma(A)\subset\Delta\}$ such that $f(T)=T$.

Then T is the unique solution in X to the operator equation $f(A)=A$. Moreover, T must be normal.

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Proof Suppose $f(z)=\sum_{n=0}^{\infty} B_n z^n$ for $z\in\Delta$. Then by the hypothesis, $\{B_n\}$ is a sequence of normal operators commuting pairwise. Let $\mathscr{B}$ denote the C*-algebra generated by $\{I, B_0, B_1, B_2, \cdots\}$. We show, first, that if there exists a B in $\mathscr{B}$ such that $\|B\|<1$ and $f(B)=B$, then $B=T$. Define for z in Δ

$$\psi(z)=(z+B)(I+B^*z)^{-1},\ \varphi(z)=(z-B)(I-B^*z)^{-1},\ F(z)=\varphi\circ f\circ\psi(z).$$

By Lemma 5.2 in [1], we have $\|\psi(z)\|<1$, $\|\varphi(z)\|<1$ for z in Δ. Since B commutes with f, it follows from Theorem 3.1 in [1] that for every z in Δ

$$\|f\circ\psi(z)\|=\|f(\psi(z))\|<1$$

and

$$\|F(z)\|=\|\varphi(f(\psi(z)))\|<1.$$

Clearly, $F(0)=0$. Therefore F takes on the form $F(z)=zh(z)$ where h is an element of $\mathscr{N}_H(\Delta)$ and $\|h(z)\|\leqslant 1$ for all z in Δ. We claim that 1 is not in $\sigma(F'(0))$. Let $\mathfrak{M}$ be the maximal ideal space of $\mathscr{B}$. If not so, then there exists an $m\in\mathfrak{M}$ such that $1=F'(0)(m)=h(0)(m)=1$. Since $h(z)(m)$ is a scalar function analytic on Δ and $|h(z)(m)|\leqslant\|h(z)\|\leqslant 1$, it follows from the maximum modulus principle that $h(z)(m)=1$, and hence $F(z)(m)=z$ for $z\in\Delta$. By the Gelfand representation theorem for commutative C*-algebras or Lemma 3 in [2], we have

$$F(z)(m)=\varphi\circ f\circ\psi(z)(m)=f(z)(m)$$

and hence $f(z)(m)=z$, in particular, $f'(0)(m)=1$, which contradicts the hypothesis: $1\bar{\in}\sigma(f'(0))$. Thus $1\bar{\in}\sigma(F'(0))$. Now we show $B=T$. An application of theorem 4 in [2] shows that in $X_1=\{A\in\mathscr{B}(H): A$ commutes with F and $\sigma(A)\subset\Delta\}$, the operator equation $F(A)=A$ has no solution other than $A=0$. Put $S=(T-B)(I-B^*T)^{-1}$. It is clear that S is in X_1, and by Lemma 2.5 in [1],

$$F(S)=\varphi\circ f\circ\psi(s)=\varphi\circ f(T)=\varphi(T)=S.$$

Thus $S=0$, i.e., $B=T$.

It remains to show the existence of a B in $\mathscr{B}$ so that $f(B)=B$. Since T co commutes with f, or $TB_n=B_nT$ for all $n=0,1,2,\cdots$, we have by Fuglede's theorem $TB_n^*=B_n^*T$ for $n=0,1,2,\cdots$ Let $\mathscr{B}_1$ represent the commutative Banach algebra generated by $\{I, T, B_n, B_n^*, n=0,1,2,\cdots\}$, and $\mathfrak{M}_1$ the maximal space of $\mathscr{B}_1$ Since $f(T)=T$, it follows from Lemma 3 in [2] that

$$T(M)=\sum_{n=0}^{\infty} B_n(M)T^n(M) \text{ for } M\in\mathfrak{M}_1.$$

It is well-known that $\sigma(B_n)=\sigma(B_n,\mathscr{B})$, and consequently

$$\sigma(B_n)=\sigma(B_n,\mathscr{B}_1)=\sigma(B_n,\mathscr{B}) \text{ for all } n=0,1,2,\cdots,$$

where $\sigma(B_n,\mathscr{B})$ and $\sigma(B_n,\mathscr{B}_1)$ denote the spectrum of B_n relative to $\mathscr{B}$ and $\mathscr{B}_1$ respectively. Define a mapping $\tau:\mathfrak{M}_1\mapsto\mathfrak{M}$ by $\tau(M)=M\cap\mathscr{B}$. It is readily seen that $B(M)=B(\tau(M))$ for any M in $\mathfrak{M}_1$ and any B in $\mathscr{B}$, and hence τ is continuous. We claim that τ is a one-to-one mapping of $\mathfrak{M}_1$ onto $\mathfrak{M}$. Assume, to the contrary, that there exist two distinct members M and M' in $\mathfrak{M}_1$ such that $B(M)=B(M')$ for all B in $\mathscr{B}$. Then $T(M)\neq T(M')$ for otherwise M and M' m must be one and the same. Write $a_n=B_n(M)=B_n(M')$ $(\forall_n\geq 0)$, and put

$$g(z)=\sum_{n=0}^{\infty}a_nz^n \text{ for } z\in\Delta.$$

Now that g is a scalar function analytic on Δ with $|g(z)|=|f(z)(M)|\leq\|f(z)\|<1$, and $\lambda_1=T(M)$ and $\lambda_2=T(M')$ are different fixed points of g in Δ, we derive from the maximum modulus principle that g is of the form $g(z)=z$. Thus $1=B_1(M)\in\sigma(B_1)$, i.e., $1\in\sigma(f'(0))$, a contradiction. Therefore τ is really a one-to-one mapping. Next, we check that τ is surjective and hence homeomor phic. Suppose this is false, namely, $\tau(\mathfrak{M}_1)\neq\mathfrak{M}$. Since $\mathfrak{M}_1$ is a compact Haus-dorff space and hence $\tau(\mathfrak{M}_1)$ is compact in $\mathfrak{M}$, it follows from the Gelfand representation theorem that there exists an element B in $\mathscr{B}$ such that $B\neq 0$, but $B(\tau(M))=0$ for all M in $\mathfrak{M}_1$. Obsreve that $B(\tau(M))=B(M)$ for every M in $\mathfrak{M}_1$ and that $\sigma(B,\mathscr{B}_1)=\sigma(B)$. Thus $\sigma(B)=\{0\}$. By normality of B, we have $B=0$, a contadiction. Therefore τ is a homeomorphiism of $\mathfrak{M}_1$ onto $\mathfrak{M}$. Then $T(\tau^{-1}(m))$ is a continuous function on $\mathfrak{M}$, and by the Gelfand representation theorem for commutative C*-algebras there exists an element B in $\mathscr{B}$ such that $B(m)=T(\tau^{-1}(m))$ for m in $\mathfrak{M}$. Applying Lemma 3 in [2], we obtain for $M\in\mathfrak{M}_1$

$$B(M)=T(M)=f(T)(M)=\sum_{n=0}^{\infty}B_n(M)T^n(M)=\sum_{n=0}^{\infty}B_n(M)B^n(M)=f(B)(M).$$

Then $B=f(B)$ follows from the fact that B and $f(B)$ are both members of $\mathscr{B}$. The proof is complete.

Corollary 1 Let f be a scalar function analytic on $\Delta=\{z:|z|<1\}$ with $|f(z)|<1$ for z in Δ, and let $X=\{A\in\mathscr{B}(H):\sigma(A)\subset\Delta\}$. Then

(i). There exists a unique operator T in X such that $f(T)=T$ if and only if $f'(0)\neq 1$ and f has a unique fixed point λ_0 in Δ. In this case, $T=\lambda_0 I$.

(ii) There exist two distinct operators S and T in X such that $f(s)=s$, f $f(T)=T$ if and only if $f(z)=z$ for all z in Δ.

Proof Clear.

For the meaning of the relative notations in the following corollary, the rea der is referred to [1](296, 305).

Corollary 2 Let $\Pi=\{z:\mathrm{Re}z>0\}$. Suppose $f\in\mathscr{N}_H(\Pi)$ satisfying the following conditions: (a). $f^*(z)+f(z)\geq 0$ for each z in Π, where $f^*(z)$ is the adjoint operator of $f(z)$; (b). $(f(1)+I)-4f'(1)$ is invertible. If there exists a an operator T in $X=\{A\in\mathscr{B}(H):A$ commutes with f and $\sigma(A)\subset\Pi\}$ such that $f(T)$

$=T$, then T is the unique solution in X to the operator equation $f(A)=A$. Moreover, T must be normal.

Proof. Let $\varphi(u)=(1+u)(1-u)^{-1}$ for $u\in\Delta$. Noting that $\varphi(\Delta)=\Pi$, one may see that $f(\varphi(u))$ is analytic on Δ. The assumption that $f(z)$ is normal and $f^*(z)+f(z)\geqslant 0$ on Π implies that $f(\varphi(u))+I$ is invertible for each u in Δ. Define

$$F(u)=(f(\varphi(u))-I)(f(\varphi(u))+I)^{-1}\quad \text{for } u\in\Delta.$$

By Lemmas 2.1, 2.3, 5.1 in [1], one may easily verify that F satisfies all the requirements of the above theorem and $S=(T-I)(T+I)^{-1}$ is a solution to the equation $F(A)=A$ in $X_1=\{A\in B(H):\sigma(A)\subset\Delta$ and A commutes with $F\}$. Since S is the only solution in X_1 to the equation $F(A)=A$ and is normal by the theorem above, the desired assertion follows.

Remark In general, a function $f\in\mathcal{N}_H(\Delta)$ with $|f(z)|<1$ for z in Δ and $1\in\sigma(f'(0))$ may possibly have a unique operator T in $X=\{A\in\mathcal{B}(H): A$ commutes with f and $\sigma(T)\subset\Delta\}$ such that $f(T)=T$.

Example Let H be a complex Hilbert space with a orthonormal basis $\{e_n: n=1,2,\cdots\}$, and let $\{E_n\}$ be the sequence of projections defined by

$$E_n(x)=\xi_n e_n \text{ for } x=\sum_{k=1}^{\infty}\xi_k e_k\in H.$$

Put

$$f(z)=\sum_{n=2}^{\infty}\left(1-\frac{1}{n+1}\right)E_n z+\frac{1}{2}E_1 z^2 \text{ for } z\in\Delta.$$

Write $B=\sum_{n=2}^{\infty}\left(1-\frac{1}{n+1}\right)E_n$. It is clear that $\|f(z)\|<1$ and $f'(0)=B$, $1\in\sigma(B)$.

Now we check that $T=0$ is the unique operator in X satisfying the operator equation $f(A)=A$. Suppose $f(T)=T$, that is

$$T=\frac{1}{2}E_1T^2+BT.$$

Since $TE_1=E_1T$, we have

$$Te_1=\frac{1}{2}E_1T^2e_1+BTE_1e_1=\frac{1}{2}T^2e_1+BE_1Te_1=\frac{1}{2}T^2e_1,\text{ or } T^2e_1=2Te_1.$$

By induction, one may infer that $T^ne_1=2^{n-1}Te_1$ for all positive integer n. Assume $\|Te_1\|>0$. Then

$$2^{\frac{n-1}{n}}\|Te_1\|^{\frac{1}{n}}\leqslant\|T^n\|^{\frac{1}{n}}.$$

By letting $n\to\infty$, we obtain $2\leqslant\lim\|T^n\|^{\frac{1}{n}}<1$, a contradiction. Thus $Te_1=0$. As for $i>1$, by the hypoyhesis: $TB=BT$, we have

$$Te_i=BTe_i=TBe_i=\left(1-\frac{1}{i+1}\right)Te_i,$$

and hence $Te_i=0$. Thus $T=0$.

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关于算子方程 $f(A)=A$ 解的唯一性

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摘要 设H是复Hilbert空间，又设 $f(z)=\sum_{n=0}^{\infty}B_n z^n$，$z\in\Delta=\{z:|z|<1\}$，其中 $\{B_n\}$ 是H上一列两两交换的正常算子，满足条件：级数按范数收敛，$\|f(z)\|<1$ 在Δ上处处成立，且$1\notin\sigma(B_1)$ 又记 $X=\{A\in\mathscr{B}(H):\sigma(A)\subset\Delta$ 且A与每个B_n交换$\}$．本文证明了，若有$T\in X$使得$f(T)=T$，则T是X中满足所论方程的唯一元素．此外，T必须是正常算子．