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Sufficient and Necessary Condition for a Pseudo-Symmetric Point Set*

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The concept of an E^n-pseudo-symmetric point set was initated in the paper [1]. The definition is as follow:

Definition Let $\mathfrak{G}$ denote a point set in E^n. $\mathfrak{G}$ is called an E^n-Pseudo-Symmetric point set if the dimention of the convex hull of $\mathfrak{G}$ is n and $\mathfrak{G}$ satisfies the following three conditions:

(i) All points of $\mathfrak{G}$ are distributed on a sphere $S^{n-1}(R)\subset E^n$;

(ii) The barycenter of $\mathfrak{G}$ is identical with the center of the hypersphere $S^{n-1}(R)$;

(iii) The inertia ellipsoid of $\mathfrak{G}$ with respect to its barycenter is a sphere.

The paper here will give a geometric inequality pertaining to an E^n-pseudo-symmetric point set. That is

Theorem Let $\mathfrak{G}=\{P_1,P_2,\cdots,P_N\}\subset S^{n-1}(R)\subset E^n$ $(N>n)$ be a finite point set, $a_{ij}=d(P_i,P_j)$ $(i,j=1,2,\cdots,N)$. Then the following inequality holds

$$\sum_{1<i<j<k<N} a_{ij}^2\cdot a_{jk}^2\cdot a_{ki}^2\leqslant\frac{4N^3(n^2-1)R^6}{3n^2} \tag{1}$$

And the equality holds in (1) if and only if $\mathfrak{G}$ is an E^n-pseudo-symmetric point set.

For the proof of the theorem we need two lemmas:

Lemma 1 Suppose $\mathfrak{G}=\{P_1,P_2,\cdots,P_N\}\subset S^{n-1}(R)\subset E^n$ $(N>n)$. Then there is only one positive eigenvalue among the eigenvalues of the squared distance matrix $A(a_{ij}^2)$ of $\mathfrak{G}$, and the positive eigenvalue is equal to the invers sum of the remainder negative eigenvalues.

Proof Since $a_{ij}^2=|P_i-P_j|^2=2R^2-2P_i\cdot P_j$, let $F=(2P_i\cdot P_j)$ and J denote the matrix in which all entries are 1.

Then

$$F=2R^2J-A.$$

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The following facts are obvious:

1. The matrix $2R^2J$ has only one non-zero positive eigenvalue $2R^2N$;
2. The matrix F is a positive semi-definite matrix;
3. The rank of the matrix A is $n+1$[2].

Arranging the egienvalues of the matrices A, F and $2R^2J$ respectively according to descending order and making use of the Weyl's Theorem (the relationship of eigenvalues)[6] it gives that

$$\lambda_i(A)+\lambda_N(F)\leqslant\lambda_i(2R^2J) \quad (i=1,2,\cdots,N).$$

Since $\lambda_N(F)=0$ because of the fact 2 and $N>n$, then

$$\lambda_1(A)\leqslant\lambda_1(2R^2J)=2R^2N, \quad \lambda_j(A)\leqslant\lambda_j(2R^2J)=0. \ (j>1).$$

Making use of the fact that $\mathrm{tr}(A)=0$ we have

$$\lambda_1(A)=-\sum_{j>1}\lambda_j(A).$$

By the fact 3 rank $(A)=n+1$, then there are n negative eigenvalues of the matrix $A=(a_{ij}^2)$. So

$$\lambda_1(A)=-\sum_{j=2}^{n+1}\lambda_j(A).$$

Lemma 2 Let $\mathfrak{C}=\{P_1,P_2,\cdots,P_N\}\subset S^{n-1}(R)\subset E^n$ $(N>n)$, $a_{ij}=d(P_i,P_j)$ $(i,j=1,2,\cdots,N)$, the matrices A, F and J as above. According to descending order arrange the eigenvalues of the matrices A and F respectively:

$$\lambda_0,0,\cdots,0,-\lambda_n,-\lambda_{n-1},\cdots,-\lambda_1; \quad \mu_1,\mu_2,\cdots,\mu_n,0,\cdots,0.$$

Then

$$\lambda_i\leqslant\mu_i \quad (i=1,2,\cdots,n). \tag{3}$$

and the equalitices hold if and only if the barycenter of $\mathfrak{C}$ is identical with the center of the hypersphere $S^{n-1}(R)$.

Proof. Because $F-(A)=2R^2J$ and the matrix $2R^2J$ is a semipositive matrix, Again by Weyl's Theorem we have

$$\lambda_i\leqslant\mu_i \quad (i=1,2,\cdots,n).$$

Now let's consider the condition for the equalitices.

Let σ_k and s_k denote the kth elementary symmetric polynomial and the sum of the kth power of the eigenvalues of a matrix respectivly. By calculating the sums of the second order principal minors of the matrices A and F respectively and making use of the Vieta's Theorem we can obtain that

$$\sigma_2(\lambda_0,-\lambda_1,\cdots,-\lambda_n)=-\lambda_0\sigma_1(\lambda_1,\lambda_2,\cdots,\lambda_n)+\sigma_2(\lambda_1,\lambda_2,\cdots,\lambda_n).$$

Notice that $\lambda_0=\sum_{i=1}^{n}\lambda_i$, then

$$-\sigma_2(\lambda_0,-\lambda_1,\cdots,-\lambda_n)=\sigma_1^2(\lambda_1,\lambda_2,\cdots,\lambda_n)-\sigma_2(\lambda_1,\lambda_2,\cdots,\lambda_n)$$

On the other hand, by (3) we have

$$-\sigma_2(\lambda_0,-\lambda_1,\cdots,-\lambda_n)\leqslant\sigma_1^2(\mu_1,\mu_2,\cdots,\mu_n)-\sigma_2(\mu_1,\mu_2,\cdots,\mu_n). \tag{4}$$

Because $F=(2R^2-a_{ij}^2)_{N\times N}$, if we denote some kth order principal minor of F by $|(2R^2-a_{ij}^2)^k|$ and of A by $|(a_{ij}^2)^k|$, then

$$|(2R^2-a_{ij}^2)^k|=(-1)^k|(a_{ij}^2)^k|+(-1)^{k-1}2R^2\sum_{s,t}(a_{ij}^2)_{st}^k,$$

where $(a_{ij}^2)_{st}^k$ denotes the cofacter of some kth order principal minor $|(a_{ij}^2)^k|$ of the matrix A, so

$$\sum_{s,t}(a_{ij}^2)_{st}^k=-\begin{vmatrix} 0 & 1 & 1 & \cdots\cdots & 1 \\ 1 & & & & \\ 1 & & (a_{ij}^2)^k & & \\ \vdots & & & & \\ 1 & & & & \end{vmatrix}$$

where the determinant on the right is called Cayley-Menger determinant pertaining to $(a_{ij}^2)^k$. By its geometric meaning[3,4,5], we have

$$\sum_{s,t}(a_{ij}^2)_{st}^k=(-1)^{k+1}2^{k-1}((k-1)!)^2\cdot V_{k-1}^2\ ,$$

where V_{k-1} denote the volume of the $(k-1)$-dimension simplex formed by some k points of $\mathfrak{E}$.

Let $n=2$, we have

$$\sigma_2(\mu_1,\mu_2,\cdots,\mu_n)=\sigma_2(\lambda_0,-\lambda_1,\cdots,-\lambda_n)+2R^2\cdot 2N_1\ , \qquad (5)$$

where the $N_1=\sum_{i<j}a_{ij}^2$.

By (4) and (5) it follows that

$$\sigma_1^2(\mu_1,\mu_2,\cdots,\mu_n)\geqslant 4R^2N_1\ . \qquad (6)$$

Because $\sigma_1(\mu_1,\mu_2,\cdots,\mu_n)=\mathrm{tr}(F)=2NR^2$, then

$$\sum_{i<j}a_{ij}^2\leqslant N^2R^2. \qquad (7)$$

This implies that the inequality (3) is equivelent to (7).

The inequality (7) is a well-known inequality and the equality holds if and only if the barycenter of $\mathfrak{E}$ is identical with the center of the hypershpere $S^{n-1}(R)$.

Proof for the theorem

For the matrix $A=(a_{ij}^2)$ we make use of the Vieta's Theorem to the eigenvalue polynomial of the matrix A. It follows that

$$\sum_{1<i<j<k<N}a_{ij}^2a_{jk}^2a_{ki}^2=\frac{1}{2}\sigma_3(\lambda_0,-\lambda_1,\cdots,-\lambda_n). \qquad (8)$$

Notice that $\mathrm{tr}(A)=\sigma_1(\lambda_0,-\lambda_1,\cdots,-\lambda_n)=s_1=0$, then

$$\sigma_3(\lambda_0,-\lambda_1,\cdots,-\lambda_n)=\frac{1}{3}s_3(\lambda_0,-\lambda_1,\cdots,-\lambda_n).$$

So

$$\sum_{1<i<j<k<N}a_{ij}^2a_{jk}^2a_{ki}^2=\frac{1}{6}s_3(\lambda_0,-\lambda_1,\cdots,-\lambda_n). \qquad (9)$$

Since

$$s_3=\lambda_0^3+(-\lambda_1)^3+\cdots+(-\lambda_n)^3=(\lambda_1+\lambda_2+\cdots+\lambda_n)^3-(\lambda_1^3+\lambda_2^3+\cdots+\lambda_n^3).$$

by the well-known inequality

$$\lambda_1^3+\lambda_2^3+\cdots+\lambda_n^3\geqslant\frac{1}{n^2}(\lambda_1+\lambda_2+\cdots+\lambda_n)^3. \tag{10}$$

So

$$s_3\leqslant\frac{n^2-1}{n^2}(\lambda_1+\lambda_2+\cdots+\lambda_n)^3. \tag{11}$$

Again by lemma 2 $\lambda_i\leqslant\mu_i$ $(i=1,2,\cdots,n)$ we have

$$s_3\leqslant\frac{n^2-1}{n^2}(\mu_1+\mu_2+\cdots+\mu_n)^3=\frac{n^2-1}{n^2}(2NR^2)^3=\frac{8N^3(n^2-1)R^6}{n^2} \tag{12}$$

Combinating (12) and (9) we finally have gotten what we want.

The equlity holds in (1) if and only if the equalities hold in (11) and (12), subsequentiy the equalities in (10) and (3) hold. That implies

$$\lambda_i=\lambda_j,\ \mu_i=\mu_j\ (i\neq j);\ \lambda_i=\mu_j\ (i=1,2,\cdots,n).$$

In other words, the barycenter of the point set $\mathfrak{S}$ is indentical with the conter of the hypersphere $S^{n-1}(R)$ and the inertia ellipsoid of $\mathfrak{S}$ is a sphere.Then the point set $\mathfrak{S}$ is an E^n-pseudo-symmetric point set.

References

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伪对称集的一个充分必要条件

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摘要 本文得到了共球诸点为E^n—伪对称集的一个充分条要条件. 若设 $\mathfrak{S}=\{P_1,P_2,\cdots,P_N\}\subset S^{n-1}(R)\subset E^n(N>n)$，$a_{ij}=d(P_i,P_j)(i,j=1,2,\cdots,N)$，则有

$$\sum_{1\leqslant i<j<k\leqslant N}a_{ij}^2a_{jk}^2a_{ki}^2\leqslant\frac{4N^3(n^2-1)R^6}{3n^2}$$

且等号成立的充分必要条件是点集 $\mathfrak{S}$ 为E^n— 伪对称集.