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Spectral Inclusion Theorem for Toeplitz Products*

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The spectral inclusion theorem for Toeplitz operator ([1], problem 196) plays a important role in studying problems of single Toeplitz operator. In this paper, we present the spectral inclusion theorem for Toeplitz products and give several corollaries.

The notation and terminology in this paper are the same as in [1]. For example, L_φ denotes Laurent operator on L^2 determined by $\varphi\in L^\infty$ (L^2 and L^∞ are both with respect to normalized Lebesgue measure on unit circle T) and T_φ denotes Toeplitz operator on Hardy space H^2, which is a closed subspace of L^2, induced by L_φ.

Theorem If $\varphi_i\in L^\infty$, $i=1,2,\cdots,k$, then

$$\Pi(L_{\varphi_1},\cdots,L_{\varphi_k})\subset\Pi(T_{\varphi_1},\cdots,T_{\varphi_k}),$$

where $\Pi(A)$ is the approximate point spectrum of operator A, K is any positive integer.

Proof Let W be the bilateral shift operator on L^2, P be the projection from L^2 onto H^2, then

$$W^{*n}PW^n\to I \qquad (n\to\infty)$$

and
$$W^{*n}T_{\varphi_i}PW^n\to L_{\varphi_i} \quad (n\to\infty)$$

in the strong operator toplogy ([1] solution 196) for $i=1,2,\cdots,k$. Therefor

$$W^{*n}T_{\varphi_1}\cdots T_{\varphi_k}PW^n=(W^{*n}T_{\varphi_1}PW^n)\cdots(W^{*n}T_{\varphi_k}PW^n)\to L_{\varphi_1}\cdots L_{\varphi_k} \quad (n\to\infty).$$

strongly. Furthermore, for any fixed complex unmber λ, we have

$$W^{*n}(T_{\varphi_1}\cdots T_{\varphi_k}-\lambda)PW^n=W^{*n}T_{\varphi_1}\cdots T_{\varphi_k}PW^n-\lambda W^{*n}PW^n\to L_{\varphi_1}\cdots L_{\varphi_k}-\lambda \ (n\to\infty)$$

strongly.

Suppose $\lambda\in\Pi(L_{\varphi_1}\cdots L_{\varphi_k})$, for every positive integer m there exists a unit vector $f_m\in L^2$ such that

$$\|(L_{\varphi_1}\cdots L_{\varphi_k}-\lambda)f_m\|<\frac{1}{m}.$$

Since, when $n\longrightarrow\infty$

$$W^{*n}PW^nf_m\longrightarrow f_m$$

and
$$W^{*n}(T_{\varphi_1}\cdots T_{\varphi_k}-\lambda)PW^nf_m\to(L_{\varphi_1}\cdots L_{\varphi_k}-\lambda)f_m$$

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in L^2, and W^* is an isometry, we can choose positive integer n_m such that

$$\|PW^{n_m}f_m\| > 1 - \frac{1}{m} \quad \text{and} \quad \|(T_{\varphi_1}\cdots T_{\varphi_k} - \lambda)PW^{n_m}f_m\| < \frac{1}{m}$$

Let

$$g_m = PW^{n_m}f_m / \|PW^{n_m}f_m\|$$

$m = 1, 2, \cdots$, then $\{g_m\}$ is a sequence of unit vectors in H^2 and

$$\|(T_{\varphi_1}\cdots T_{\varphi_k} - \lambda)g_m\| < \frac{2}{m},$$

for al l $m \geqslant 2$. It follows that $\lambda \in \Pi(T_{\varphi_1} T_{\varphi_k})$.

Corollary 1 $\Lambda(L_{\varphi_1}\cdots L_{\varphi_k}) \subset \Lambda(T_{\varphi_1}\cdots T_{\varphi_k})$, where $\Lambda(A)$ is the spectrum of operator A.

Proof Since $L_{\varphi_1}\cdots L_{\varphi_k} = L_{\varphi_1\cdots\varphi_k}$ and spectrum of a Laurent operator coincide with its approximate point spectrum ([1], solution 66).

Corollary 2 $r(T_{\varphi_1}\cdots T_{\varphi_k}) \geqslant \|\varphi_1\cdots\varphi_k\|_\infty$, where $r(A)$ is the spectral radius of operator A.

Proof Immediately follows from Corollary 1.

Note The inequality in Corollary 2 can be strict. For example, let

$$\varphi_1(e^{i\theta}) = \chi_{[0,\pi]}(\theta), \quad \varphi_2(e^{i\theta}) = \chi_{[\pi,2\pi]}(\theta), \quad \theta \in [0, 2\pi]$$

then $\|\varphi_1\varphi_2\| = 0$ and by applying spectral mapping theorem, we have

$$\Lambda(T_{\varphi_1}T_{\varphi_2}) = \Lambda(T_{\varphi_1} - (T_{\varphi_1})^2) = [0, 1/4];$$
$$r(T_{\varphi_1}T_{\varphi_2}) = 1/4.$$

because $\Lambda(T_{\varphi_1}) = [0, 1]$(see [1] solution 198).

At iast, we extend the zero-divisors problem ([1] problem 195) for Toeplitz operators to the situation of having three factors.

Corollary 3 A necessary and suficient condition that the products of three Toeplitz operators be zero is that at least one factor be zero.

Proof Observe first that, for a nonzero Toeplitz operator T_φ, $\mathrm{Ker}(T_\varphi) \neq \{0\}$ only if $\varphi \neq 0$ almost everywhere. In fact, if $f \in H^2$, $f \neq 0$ and $T_\varphi f = P(\varphi f) = 0$, then $(\varphi f)^* = \varphi^* f^* \in H^2$ and $\varphi^* f^* \neq 0$. From F. and M. Riesz Theorem, f and $\varphi^* f^*$ don't vanish almost everywhere. Therefor, $\varphi \neq 0$ almost everywhere.

Assume T_{φ_i}, $i = 1, 2, 3$, be Toeplitz operators and $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3} = 0$, so $T_{\varphi_3^*}T_{\varphi_2^*}T_{\varphi_1^*} = 0$. If $T_{\varphi_i} \neq 0$, $i = 1, 2, 3$, then $T_{\varphi_2}T_{\varphi_3} \neq 0$ and $T_{\varphi_2^*}T_{\varphi_1^*} \neq 0$, and therefore $\mathrm{Ker}(T_{\varphi_1}) \neq \{0\}$, $\mathrm{Ker}(T_{\varphi_3^*}) \neq \{0\}$. By the asseriton just proved, φ_1 and φ_3 don't vanish almost everywhere. From Corollary 3, $\varphi_1\varphi_2\varphi_3 = 0$, thus $\varphi_2 = 0$ almost everywhere, and $T_{\varphi_2} = 0$, a contra diction.

Reference

[1] P.R. Halmos, A Hilbert Space Problem Book, Von Nostrand Co. Princeton, 1967.