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On Certain Classes of Univalent Functions with Negative and Missing Coefficients

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Abstract Let $p^*(A,B,k,\alpha,\beta)$ be the class of functions $f(z)=a_1-\sum_{n=k}^{\infty}|a_n|z^n$ $(a_1>0, k\geqslant 2)$ regular and univalent in the unit disc $U=\{z: |z|<1\}$ and satisfying

$$\left|\frac{f'(z)-a_1}{[A+(B-A)\alpha\beta]a_1-[A+(B-A)\beta]f'(z)}\right|<1, z\in U,$$

where $-1\leqslant B<A\leqslant 1$, $-1\leqslant B<0$, $0\leqslant\alpha<1$ and $0<\beta\leqslant 1$. We denote by $p_0(A,B,k,\alpha,\beta,z_0)$ and $p_1(A,B,k,\alpha,\beta,z_0)$, two subclasses of $p^*(A,B,k,\alpha,\beta)$, consisting of the functions which satisfy $f(z_0)=z_0$ and $f'(z_0)=1$, respectively, here $0<z_0<1$. In this paper we obtain coefficient estimates, distortion and closure theorems and radius of convexity of order ρ $(0\leqslant\rho<1)$ for the classes $p_0(A,B,k,\alpha,\beta,z_0)$ and $p_1(A,B,k,\alpha,\beta,z_0)$.

I. Introduction.

Let $P^*(A,B,k,\alpha,\beta)$ be the class of functions $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$ $(a_1>0, k\geqslant 2)$ regular and univalent in the unit disc $U=\{z: |z|<1\}$, and satisfying the condition

$$\left|\frac{f'(z)-a_1}{[A+(B-A)\alpha\beta]a_1-[A+(B-A)\beta]f'(z)}\right|<1, z\in U, \qquad (1.1)$$

where $-1\leqslant B<A\leqslant 1$, $-1\leqslant B<0$, $0\leqslant\alpha<1$ and $0<\beta\leqslant 1$. Let $0<z_0<1$. we introduce two subclasses, namely $P_0(A,B,k,\alpha,\beta,z_0)$ and $P_1(A,B,k,\alpha,\beta,z_0)$, of $P^*(A,B,k,\alpha,\beta)$. A function f of $P^*(A,B,k,\alpha,\beta)$ belongs to $P_0(A,B,k,\alpha,\beta,z_0)$ if $f(z_0)=z_0$, whereas $P_1(A,B,k,\alpha,\beta,z_0)$ is the class of those functions of $P^*(A,B,k,\alpha,\beta)$ which satisfy $f'(z_0)=1$.

In [3] Kumar and Shukla obtained coefficient estimates, distortion and closure theorems and radius of convexity of order ρ $(0\leqslant\rho<1)$ for the classes $P_0(A,B,k,0,1,z_0)$ and $P_1(A,B,k,0,1,z_0)$. Also Kumar [2] has obtained many results including coefficient estimates, distortion theorems, radius of convexity and class preserving integral operators for the class $P^*(A,B,k,0,1)$ when $a_1=1$.

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In the present paper we obtained coefficient estimates, distortion and covering theorems and radius of convexity of order $\rho\,(0\leqslant\rho<1)$ for the classes $P_0(A, B, k, a, \beta, z_0)$ and $P_1(A, B, k, a, \beta, z_0)$. Further, it is shown that these classes are closed under "arithmetic mean" and "convex linear combinations".

2. Coefficient estimates

Theorem 1 If $f(z)=a_1 z-\sum_{n=k}^{\infty}|a_n|z^n$ is analytic in U and satisfies $f(z_0)=z_0$. Then $f\in P_0(A, B, k, a, \beta, z_0)$ if and only if

$$\sum_{n=k}^{\infty}\{n(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{n-1}\}\,|a_n|\leqslant(A-B)\beta(1-a). \quad (2.1)$$

The result is sharp.

Proof Since $f(z_0)=z_0$, we have

$$a_1=1+\sum_{n=k}^{\infty}|a_n|z_0^{n-1}. \quad (2.2)$$

Therefore, our theorem is proved if we show that $f\in P^*(A,B,k,a,\beta)$ if and only if

$$\sum_{n=k}^{\infty}n(1-A-(B-A))\,|a_n|\leqslant(A-B)\beta(1-a)a_1. \quad (2.3)$$

Suppose that the inequality (2.3) holds. Let $|z|=1$. Then

$$|f'(z)-a_1|-|[A+(B-A)a\beta]a_1-[A+(B-A)\beta]f'(z)|$$

$$=\left|-\sum_{n=k}^{\infty}n\,|a_n|z^{n-1}\right|-\left|(A-B)\beta(1-a)a_1+[A+(B-A)\beta]\sum_{n=k}^{\infty}n\,|a_n|z^{n-1}\right|$$

$$\leqslant\sum_{n=k}^{\infty}\{n(1-A-(B-A)\beta)\,|a_n|\}-(A-B)\beta(1-a)a_1\leqslant 0.$$

(since $-1\leqslant B<A\leqslant 1$, $-1\leqslant B\leqslant 0$, $0\leqslant a<1$, $0\leqslant\beta<1$ and $(B-A)\beta<0$).

Hence, by maximum modulus principle, $f\in P^*(A, B, k, a, \beta)$.

In order to prove the converse, let $f\in P^*(A, B, k, a, \beta)$. It follows, therefore, that

$$\left|\frac{f'(z)-a_1}{[A+(B-A)a\beta]a_1-[A+(B-A)\beta]f'(z)}\right|$$

$$=\left|\frac{-\sum_{n=k}^{\infty}n\,|a_n|z^{n-1}}{(A-B)\beta(1-a)a_1+[A+(B-A)\beta]\sum_{n=k}^{\infty}n\,|a_n|z^{n-1}}\right|<1,\ z\in U.$$

Since $|Re(z)|\leqslant|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=k}^{\infty}n\,|a_n|z^{n-1}}{(A-B)\beta(1-a)a_1+[A+(B-A)\beta]\sum_{n=k}^{\infty}n\,|a_n|z^{n-1}}\right\}<1.$$

Choose values of z on the real axis so that $f'(z)$ is real. Upon clearing the denominator in this last inequality and letting $z \to 1$ through real values, we easily arrive at (2.3). The inequality (2.1) follows now by eliminating a_1 from (2.2) and (2.3).

Equality holds in (2.1) for the function

$$f(z)=\frac{n(1-A-(B-A)\beta)z-(A-B)\beta(1-\alpha)z^n}{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}}\quad (n=k,\ k+1,\cdots\cdots),$$

and hence the result is sharp.

Corollary 1 Let $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$. If $f\in P_0(A,B,k,\alpha,\beta,z_0)$, then

$$|a_n|\leqslant\frac{(A-B)\beta(1-\alpha)}{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}}.$$

Theorem 2 Let $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$. If f is regular in U and satisfies $f'(z_0)=1$, then $f\in P_1(A,B,k,\alpha,\beta,z_0)$ if and only if

$$\sum_{n=k}^{\infty}n\{(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}\}|a_n|\leqslant(A-B)\beta(1-\alpha).$$

The result is sharp.

Proof Since $f'(z_0)=1$, we have

$$a_1=1+\sum_{n=k}^{\infty}n|a_n|z_0^{n-1}. \tag{2.4}$$

Eliminating a_1 from (2.3) and (2.4) we get the required result.

Sharpness follows if we take

$$f(z)=\frac{n(1-A-(B-A)\beta)z-(A-B)\beta(1-\alpha)z^n}{n\{(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}\}}\quad (n=k,k+1,\cdots\cdots).$$

Corollary 2 Let $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$. If $f\in P_1(A,B,k,\alpha,\beta,z_0)$, then

$$|a_n|\leqslant\frac{(A-B)\beta(1-\alpha)}{n\{(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}\}}.$$

3. Distortion properties

Theorem 3 If $f\in P_0(A,B,k,\alpha,\beta,z_0)$ and $|z|=r$, then

$$ar-br^k\leqslant|f(z)|\leqslant ar+br^k \tag{3.1}$$

and

$$a-kbr^{k-1}\leqslant|f'(z)|\leqslant a+kbr^{k-1}, \tag{3.2}$$

where

$$a=\frac{k(1-A-(B-A)\beta)}{k(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{k-1}}$$

and

$$b=\frac{(A-B)\beta(1-\alpha)}{k(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{k-1}}.$$

The result is sharp.

Proof Let $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$. Then, it follows from Theorem 1 that

$$\sum_{n=k}^{\infty}|a_n|\leqslant\frac{(A-B)\beta(1-a)}{k(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}} \tag{3.3}$$

and

$$\sum_{n=k}^{\infty}n|a_n|\leqslant\frac{k(A-B)\beta(1-a)}{k(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}}. \tag{3.4}$$

Also, from the representation of $f(z)$ we have

$$|f(z)|\leqslant a_1r+\sum_{n=k}^{\infty}|a_n|r^n\leqslant a_1r+r^k\sum_{n=k}^{\infty}|a_n|,$$

$$|f(z)|\geqslant a_1r-\sum_{n=k}^{\infty}|a_n|r^n\geqslant a_1r-r^k\sum_{n=k}^{\infty}|a_n|$$

$$|f(z)|\leqslant a_1+\sum_{n=k}^{\infty}n|a_n|r^{n-1}\leqslant a_1+r^{k-1}\sum_{n=k}^{\infty}n|a_n|,$$

$$|f(z)|\geqslant a_1-\sum_{n=k}^{\infty}n|a_n|r^{n-1}\geqslant a_1-r^{k-1}\sum_{n=k}^{\infty}n|a_n|.$$

The required inequalities follow now by using (2.2), (3.3)and (3.4) in the above four inequalities.

To establish the sharpness of the result we take

$$f(z)=\frac{k(1-A-(B-A)\beta)z-(A-B)\beta(1-a)z^k}{k(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}}.$$

For this function, equality on the left hand side of (3.1) and (3.2) is attained at $z=r$, whereas, the equality on the right hand side is attained at $z=re^{i\pi/(k-1)}$. Hence the result is sharp.

Letting $r\to 1$ in (3.1) we have the following:

Corollary 3 If $f\in P_0(A,B,k,a,\beta,z_0)$, then the disc U is mapped by f onto a domain that contains the disc

$$|w|<\frac{k(1-A-(B-A)\beta)-(A-B)\beta(1-a)}{k(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}}.$$

The result is sharp.

Theorem 4 If $f\in P_1(A,B,k,a,\beta,z_0)$ and $|z|=r$, then

$$cr-dr^k\leqslant|f(z)|\leqslant cr+dr^k \tag{3.5}$$

and

$$c-dkr^{k-1}\leqslant|f'(z)|\leqslant c+dkr^{k-1}, \tag{3.6}$$

where

$$c=\frac{(1-A-(B-A)\beta)}{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}}$$

and

$$d = \frac{(A-B)\beta(1-a)}{k\{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}\}}.$$

The result is sharp.

Proof Let $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$. Then, it follows from Theorem 2 that

$$\sum_{n=k}^{\infty}|a_n| \leqslant \frac{(A-B)\beta(1-a)}{k\{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}\}}$$

and

$$\sum_{n=k}^{\infty}n|a_n| \leqslant \frac{(A-B)\beta(1-a)}{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}}.$$

The required bounds for $|f(z)|$ and $|f'(z)|$ can be obtained now in the same way as obtained in the preceding theorem.

Equality in (3.5) and (3.6) is attained if we take

$$f(z) = \frac{k(1-A-(B-A)\beta)z-(A-B)\beta(1-a)z^k}{k\{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}\}}.$$

Letting $r\to 1$ in (3.5) we have the following:

Corollary 4. If $f\in P_1(A,B,k,a,\beta,z_0)$, then the disc U is mapped by f onto a domain that contains the disc

$$|w| < \frac{k(1-A-(B-A)\beta)-(A-B)\beta(1-a)}{k\{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{k-1}\}}.$$

The result is sharp.

4. Radius of convexity

In this section we determine the radius of convexity of order ρ $(0\leqslant\rho<1)$ for the classes $P_0(A,B,k,a,\beta,z_0)$ and $P_1(A,B,k,a,\beta,z_0)$.

Theorem 5 If $f\in P_0(A,B,k,a,\beta,z_0)$ or $P_1(A,B,k,a,\beta,z_0)$, then f is convex function of order ρ in the disc $|z|<R^*$, where

$$R^* = \inf_{n>k}\left[\frac{(1-A-(B-A)\beta)(1-\rho)}{(A-B)\beta(1-a)(n-\rho)}\right]^{\frac{1}{(n-1)}}$$

This result is sharp.

Proof In order to establish the theorem, it sufficies to show that $\left|\frac{zf''(z)}{f'(z)}\right| < (1-\rho)$ holds in $|z|<R^*$.

Let $f(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n$. Then

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|-\sum_{n=k}^{\infty}n(n-1)|a_n|z^{n-1}/\left(a_1-\sum_{n=k}^{\infty}n|a_n|z^{n-1}\right)\right|$$

$$\leqslant \sum_{n=k}^{\infty}n(n-1)|a_n||z|^{n-1}/\left(a_1-\sum_{n=k}^{\infty}n|a_n||z|^{n-1}\right).$$

Therefore, $\left|\frac{zf''(z)}{f'(z)}\right|<(1-\rho)$ holds if

$$\sum_{n=k}^{\infty} n(n-\rho)|a_n||z|^{n-1}<(1-\rho)a_1. \tag{4.1}$$

when $f\in P_0(A, B, k, a, \beta, z_0)$, using (2.2) we find that the inequality (4.1) is equivalent to

$$\sum_{n=k}^{\infty} [\{n(n-\rho)|z|^{n-1}-(1-\rho)z_0^{n-1}\}|a_n|]<(1-\rho). \tag{4.2}$$

But Theorem 1 ensures

$$\sum_{n=k}^{\infty} [(1-\rho)\{\frac{n(1-A-(B-A)\beta)}{(A-B)\beta(1-a)}-z_0^{n-1}\}|a_n|]\leqslant(1-\rho).$$

Hence (4.2) holds if

$$\{n(n-\rho)|z|^{n-1}-(1-\rho)z_0^{n-1}\}|a_n|<$$

$$(1-\rho)\{\frac{n(1-A-(B-A)\beta)}{(A-B)\beta(1-a)}-z_0^{n-1}\}|a_n|, \text{ for each } n=k,\ k+1,\cdots$$

or if

$$|z|<[\frac{n(1-A-(B-A)\beta)}{(A-B)\beta(1-a)}\circ(\frac{1-\rho}{n-\rho})]^{\frac{1}{(n-1)}} \quad \text{for each } n=k,k+1,\cdots$$

Thus f is convex function of order ρ in $|z|<R^*$.

In other case when $f\in P_1(A, B, k, a, \beta, z_0)$, using (2.4) we find

$$\sum_{n=k}^{\infty} [n\{(n-\rho)|z|^{n-1}-(1-\rho)z_0^{n-1}\}|a_n|]<(1-\rho). \tag{4.3}$$

Therefore, in view of Theorem 2, the inequality (4.3) holds if

$$n\{(n-\rho)|z|^{n-1}-(1-\rho)z_0^{n-1}\}|a_n|$$

$$<n(1-\rho)\{\frac{(1-A-(B-A)\beta)}{(A-B)\beta(1-a)}-z_0^{n-1}\}|a_n|, \text{ for each } n=k+1,k+1,\cdots$$

or if

$$|z|<[\frac{(1-A-(B-A)\beta)}{(A-B)\beta(1-a)}(\frac{1-\rho}{n-\rho})]^{\frac{1}{(n-1)}}, \text{ for each } n=k,\ k+1,\cdots.$$

This completes the proof of the theorem.

Sharpness for the class $P_0(A, B, a, \beta, k, z_0)$ follows by taking

$$f(z)=\frac{n(1-A-(B-A)\beta)z-(A-B)\beta(1-a)z^n}{n(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{n-1}},$$

whereas, for the class $P_1(A, B, k, a, \beta, z_0)$, sharpness follows if we take

$$f(z)=\frac{n(1-A-(B-A)\beta)z-(A-B)\beta(1-a)z^n}{n\{(1-A-(B-A)\beta)-(A-B)\beta(1-a)z_0^{n-1}\}}.$$

5. Closure properties

In this section we show that both the classes under consideration are closed under "arithmetic mean" and "convex linear combinations".

Theorem 6 Let $f_j(z)=a_{1j}z-\sum\limits_{n=k}^{\infty}|a_{nj}|z^n$, $j=1,2,\cdots$, and $h(z)=b_1z-\sum\limits_{n=2}^{\infty}|b_n|z^n$, where $b_1=\sum\limits_{j=1}^{\infty}\lambda_j a_{1j}$, $b_n=\sum\limits_{j=1}^{\infty}\lambda_j a_{nj}$, $(n=k,\ k+1,\cdots)$, $\lambda_j\geqslant 0$ and $\sum\limits_{j=1}^{\infty}\lambda_j=1$. If $f_j\in P_0(A,B,k,\alpha,\beta,z_0)$ (or $P_1(A,B,k,\alpha,\beta,z_0)$) for each $j=1,2,\cdots$, then $h\in P_0(A,B,k,\alpha,\beta,z_0)$ (or $P_1(A,B,k,\alpha,\beta,z_0)$).

Proof If $f_j\in P_0(A,B,k,\alpha,\beta,z_0)$, then we get from Theorem 1 that

$$\sum_{n=k}^{\infty}\{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}\}|a_{nj}|$$

$$\leqslant(A-B)\beta(1-\alpha),\quad j=1,2,\cdots.$$

Therefore

$$\sum_{n=k}^{\infty}\{n(1-A-(B-A)-(A-B)\beta(1-\alpha)z_0^{n-1}\}|b_n|$$

$$\leqslant\sum_{n=k}^{\infty}[\{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}\}\{\sum_{j=1}^{\infty}\lambda_j|a_{nj}|\}]$$

$$\leqslant(A-B)\beta(1-\alpha).$$

Hence, by Theorem 1, $h\in P_0(A,B,k,\alpha,\beta,z_0)$.

Similarly, by using Theorem 2, we can prove that $h\in P_1(A,B,k,\alpha,\beta,z_0)$ if $f_j\in P_1(A,B,k,\alpha,\beta,z_0)$ for each $j=1,2,\cdots$.

Theorem 7 Let $f_1(z)=z$ and

$$f_n(z)=\frac{n(1-A-(B-A)\beta)z-(A-B)\beta(1-\alpha)z^n}{n(1-A-(B-A)\beta-(A-B)\beta(1-\alpha)z_0^{n-1}}\quad(n=k,k+1,\cdots).$$

Then $f\in P_0(A,B,k,\alpha,\beta,z_0)$ if and only if it can be expressed in the form

$$f(z)=\lambda_1f_1(z)+\sum_{n=k}^{\infty}\lambda_nf_n(z),$$

where $\lambda_n\geqslant 0$ and $\lambda_1+\sum\limits_{n=k}^{\infty}\lambda_n=1$.

Proof Let us suppose that

$$f(z)=\lambda_1f_1(z)+\sum_{n=k}^{\infty}\lambda_nf_n(z)=a_1z-\sum_{n=k}^{\infty}|a_n|z^n,$$

where

$$a_1=\lambda_1+\sum_{n=k}^{\infty}\frac{n(1-A-(B-A)\beta)\lambda_n}{n(1-A-(B-A)\beta)-(A-\beta)\beta(1-\alpha)z_0^{\alpha-1}}$$

and

$$|a_n|=\frac{(A-B)\beta(1-\alpha)\lambda_n}{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}},\quad(n=k,k+1,\cdots).$$

Then it is easy to see that $f(z_0)=z_0$ and the condition (2.1) is satisfied. Hence

$f \in P_0(A, B, k, \alpha, \beta, z_0)$.

Conversely, let $f \in P_0(A, B, k, \alpha, \beta, z_0)$ and $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. Then, by Corollary 1, we have

$$|a_n| \leqslant \frac{(A-B)\beta(1-\alpha)}{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}}, \quad (n=k, k+1, \cdots).$$

Setting

$$\lambda_n = \left[\frac{n(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}}{(A-B)\beta(1-\alpha)}\right]|a_n|, \quad (n=k, k+1, \cdots).$$

and

$$\lambda_1 = 1 - \sum_{n=k}^{\infty} \lambda_n,$$

we have

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z).$$

The proof is completed now.

The following theorem can be proved on the lines of the proof of preceding theorem. We omit the details of its proof.

Theorem 8 Let $f_1(z) = z$ and

$$f_n(z) = \frac{n(1-A-(B-A)\beta)z-(A-B)\beta(1-\alpha)z^n}{n\{(1-A-(B-A)\beta)-(A-B)\beta(1-\alpha)z_0^{n-1}\}} \quad (n=k, k+1, \cdots).$$

Then $f \in P_1(A, B, k, \alpha, \beta, z_0)$ if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geqslant 0$ and $\lambda_1 + \sum_{n=k}^{\infty} \lambda_n = 1$.

Remarks

1. Putting $\alpha = 0$ and $\beta = 1$ in the above results, we get the results obtained by Kumar and Shukla [3].

2. Putting $A = \beta$ and $B = \beta - 2\gamma$, $0 < \beta \leqslant 1$ and $\frac{1}{2} \leqslant \gamma \leqslant 1$ in the above results, we get the results obtained by Guqta and Ahmad [1].

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