Journal of Mathematical Research and Exposition
Vol.10 No.1 Feb. 1990

Equivalence between Bloch Space and BMOA Space*

Xiao Jie

(Dept. Math., Peking University)

Let Ω be a hyperbolic domain in the finite complex plane **C**, $\lambda_\Omega(z)$ be Poincaré metric with curvature -4 on Ω and $\delta_\Omega(z)$ be the Euclidean distance from point z to the boundary $\partial\Omega$ of Ω. Set $\mathrm{A}(\Omega)$, $\mathrm{H}(\Omega)$ and $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ to denote the class of analytic functions, real-valued harmonic functions and locally integrable functions on Ω respectively.

Bloch space on Ω is defined by

$$\mathrm{B}(\Omega):=\{f: f\in \mathrm{A}(\Omega),\ \|f\|_{\mathrm{B}(\Omega)}:=\sup_{z\in\Omega}\frac{|f'(z)|}{\lambda_\Omega(z)}<\infty\}.$$

Also, let $\mathrm{BH}(\Omega):=\{f: f\in \mathrm{H}(\Omega),\ \|f\|_{\mathrm{BH}(\Omega)}:=\sup_{z\in\Omega}\delta_\Omega(z)\,|\nabla f(z)|<\infty\}$, where ∇ is the gradient operator.

BMO space on Ω is given by

$$\mathrm{BMO}(\Omega,m):=\{f: f\in \mathrm{L}^1_{\mathrm{loc}}(\Omega),\ \|f\|_{m\Omega}:=\sup_{\Delta}\{\frac{1}{m(\Delta)}\int_\Delta |f(z)-f_\Delta|\,\mathrm{d}m(z)\}<\infty\}$$

where $z=x+iy$, $\mathrm{d}m(z)=\mathrm{d}x\mathrm{d}y$; $f_\Delta=\frac{1}{m(\Delta)}\int_\Delta f(z)\,\mathrm{d}m(z)$, $m(X)$ is Lebesgue measure of set X. The supremum is taken over all Euclidean disks $\Delta\subset\Omega$. Also, set $\mathrm{BMOH}(\Omega,m):=\mathrm{BMO}(\Omega,m)\cap \mathrm{H}(\Omega)$, $\mathrm{BMOA}(\Omega,m):=\mathrm{BMO}(\Omega,m)\cap A(\Omega)$.

In this paper, we have discussed the equivalence between $\mathrm{B}(\Omega)$ and $\mathrm{BMOA}(\Omega,m)$.

First, we need several lemmas:

Lemma 1 Let p be an analytic covering map of $\mathrm{D}:=\{z:|z|<1\}$ onto Ω. Then $f\in \mathrm{B}(\Omega)$ if and only if $f\circ p\in \mathrm{B}(\mathrm{D})$.

Lemma 2 If $f\in \mathrm{B}(\Omega)$ and $\|f\|_{\mathrm{B}(\Omega)}>0$, then for every Euclidean disk $\Delta\subset\Omega$, $t>0$ we have $m(\{z: z\in\Delta,\ |f(z)-f_\Delta|>t\})<4m(\Delta)\exp(-t/\|f\|_{\mathrm{B}(\Omega)})$.

Lemma 3 $f\in \mathrm{BH}(\Omega)$ if and only if $f\in \mathrm{BMOH}(\Omega,m)$ [1].

Next, we imply the main results in two cases.

Case 1: Ω is the simply connected domain.

H.M.Reimann[2] proved the following fact:

* Received Apr.6, 1988.

Lemma 4 Let Ω, $\tilde{\Omega}$ be simply connected domains, g be an one to one conformal map of $\tilde{\Omega}$ onto Ω. If $f\in \mathrm{BMO}(\Omega, m)$, then $f\circ g\in \mathrm{BMO}(\tilde{\Omega}, m)$, and more $\frac{1}{a}\|f\|_{m\Omega}<\|f\circ g\|_{m\tilde{\Omega}}<a\|f\|_{m\Omega}$, here $a>0$ is an absolute constant.

Hence we have:

Theorem 1 Let Ω be a simply connected domain. Then the following are equivalent:

(1) $f\in \mathrm{B}(\Omega)$; (2) $\mathrm{Re} f\in \mathrm{BMOH}(\Omega, m)$; (3) $f\in \mathrm{BMOA}(\Omega, m)$; (4) $\mathrm{Re} f\in \mathrm{BH}(\Omega)$.

Case 2: Ω is a multiply connected domain.

B.G.Osgood[1] showed the lemma as below:

Lemma 5 Let $C(\Omega):=\inf\limits_{z\in\Omega}\lambda_\Omega(z)\delta_\Omega(z)>0$, p be an analytic covering map of D onto Ω. If $f\in \mathrm{BMOH}(\Omega, m)$, then $f\circ p\in \mathrm{BMOH}(\mathrm{D}, m)$ and $\|f\circ p\|_{m\mathrm{D}}<b\|f\|_{m\Omega}$, $b>0$ is an absolute constant.

Thus we get another result right away:

Theorem 2 Let Ω be a multiply connected domain with $C(\Omega)>0$. Then the following are equivalent:

(1) $f\in \mathrm{B}(\Omega)$; (2) $\mathrm{Re} f\in \mathrm{BMOH}(\Omega, m)$; (3) $f\in \mathrm{BMOA}(\Omega, m)$; (4) $\mathrm{Re} f\in \mathrm{BH}(\Omega)$.

References

[1] B.G. Osgood, Indiana Univ. Math. J. Vol.31, No.4 (1982) 449—461.

[2] H. M. Reimann, Comment Math. Helv. 49 (1974) 260—274.

Bloch空间与BMOA空间的等价性

肖　杰

（北京大学数学系）

摘要　设Ω是有限复平面**C**上的双曲型区域，$\lambda_\Omega(z)$为其上的曲率为-4的Poincare度量，$\delta_\Omega(z):=\mathrm{dist}(z,\partial\Omega)$. 用$\mathrm{B}(\Omega)$，$\mathrm{BH}(\Omega)$，$\mathrm{BMOA}(\Omega, m)$和$\mathrm{BMOH}(\Omega, m)$分别表示Ω上的Bloch空间，拟梯度函数空间，解析的面积BMO空间和实值调和的面积BMO空间. 本文证明了如下结论：

定理　设Ω是**C**上的双曲型区域. 如果$C(\Omega):=\inf\limits_{z\in\Omega}\lambda_\Omega(z)\delta_\Omega(z)>0$，那么下面四条是等价的：

(1) $f\in \mathrm{B}(\Omega)$；(2) $\mathrm{Re} f\in \mathrm{BMOH}(\Omega, m)$；(3) $f\in \mathrm{BMOA}(\Omega, m)$；(4) $\mathrm{Re} f\in \mathrm{BH}(\Omega)$.