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Lipschitz Constants for the Bernstein Polynomials Defined over a Triangle*

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1. Introduction

Let T be a triangle arbitrarily given with points T_1, T_2, T_3 as its vertices. It is well known that an arbitrary point P in the plane on which triangle T lies can be expressed by its barycentric coordinates (u, v, w) with respect to T and that $u+v+w=1$ is satisfied. We identify P with its barycentric coordinates and write $P=(u, v, w)$. If $P\in T$, we have further restrictions $0\leqslant u, v, w\leqslant 1$.

Suppose $f(P)$ is an arbitrary function defined on T, then we define the Bernstein polynomials of degree n over T associated with f by

$$B^n(f, p):=\sum_{i+j+k=n} f(i/n, j/n, k/n) B^n_{i,j,k}(P) \tag{1}$$

where $B^n_{i,j,k}(P):=\frac{n!}{i!j!k}u^iv^jw^k$ are Bernstein basic functions.

Bernstein polynomial $B^n(f, P)$ preserves the properties of function $f(P)$ to a considerable degree. Recently B.M.Brown[1] proved the following result for univariate Bernstein polynomials: If $f\in \mathrm{Lip}_A\lambda$, $0<\lambda\leqslant 1$, then for all $n\geqslant 1$, $B^n(f, x)\in \mathrm{Lip}_A\lambda$. The purpose of this paper is to prove the similar result for Bernstein polynomials defined over a triangle. In the following, let $T:=\{(x, y): x\geqslant 0, y\geqslant 0, x+y\leqslant 1\}$ be a standard triangle. At first, we give the following

Definition A function $f(P)$ is Lipschitz continuous of order λ, $0<\lambda\leqslant 1$ on triangle T, if there exists constant $A>0$ such that for every pair points $P_1, P_2\in T$, we have

$$|f(P_1)-f(P_2)|\leqslant A\|P_1-P_2\|^\lambda \tag{2}$$

written by $f(P)\in \mathrm{Lip}_A\lambda$. Here the constant A, which depends upon f and λ, is known as Lipschitz constant of f and $\|\cdot\|$ denotes Euclidean norm.

2. Main Theorem

Before stating the main theorem, we need to point out two facts. If $P_i=(u_i,$

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$v_i, w_i) = u_iT_1 + v_iT_2 + w_iT_2$, $i = 1, 2$. Since $(u_2 - u_1) + (v_2 - v_1) + (w_2 - w_1) = 0$, without loss of generality, we may suppose $(u_2 - u_1)(v_2 - v_1) > 0$, using $w_2 - w_1 = -(u_2 - u_1) - (v_2 - v_1)$, we have

$$\|P_2 - P_1\|^2 = \|(u_2 - u_1)(T_1 - T_3) + (v_2 - v_1)(T_2 - T_3)\|^2$$
$$< ((u_2 - u_1)h_2 + (v_2 - v_1)h_1)^2 < 2\|P_1 - P_2\|^2 \qquad (3)$$

here $h_2 = \|T_1 - T_3\|$, $h_1 = \|T_2 - T_3\|$.

The other fact is that when $f(P)$ is a convex function over T, then we have

$$B^n(f, P) > B^{n+1}(f, P), \quad n = 1, 2, \cdots.$$

Therefore, when $f(P)$ is a concave function we have

$$B^n(f, P) < f(P) \qquad (4)$$

for all $n > 1$. Now we prove the following main

Theorem Suppose $f(P) \in \mathrm{Lip}_A\lambda$, $0 < \lambda < 1$, then for all $n > 1$, we have

$$B^n(f, P) \in \mathrm{Lip}_{\sqrt{2}^{\lambda}A}\lambda.$$

Proof We consider the following two cases:

Case 1. Suppose $u_2 > u_1$, $v_2 > v_1$.

Since $u + v + w = 1$, we can think of $B^n(f, P)$ as functions of variables u and v. Then from (1), we have

$$B^n(f, P_2) = \sum_{s=0}^{n}\sum_{t=0}^{n-s} f_{s,t}\frac{n!}{s!\,t!\,(n-s-t)!}u_2^s v_2^t(1 - u_2 - v_2)^{n-s-t} \qquad (5)$$

here $f_{s,t} := f(s/n, t/n)$. Note that

$$u_2^s v_2^t = \sum_{i=0}^{s}\binom{s}{i}u_1^i(u_2 - u_1)^{s-i}\sum_{j=0}^{t}\binom{t}{j}v_1^j(v_2 - v_1)^{t-j}. \qquad (6)$$

Combining (5) and (6), after inverting the order of summation and writing $s - i = k$, $t - j = r$, then we have

$$B^n(f, P_2) = \sum_{i+j+k+r<n} f\left(\frac{i+k}{n}, \frac{j+r}{n}\right)\frac{n!}{i!\,j!\,k!\,r!\,(n-i-j-r-k)!} \qquad (7)$$
$$\cdot u_1^i v_1^j(u_2 - u_1)^k(v_2 - v_1)^r(1 - u_2 - v_2)^{n-i-j-r-k}.$$

In the other hand, by definition

$$B^n(f, P_1) = \sum_{i+j<n} f_{i,j}u_1^i v_1^j(1 - u_1 - v_1)^{n-i-j}\frac{n!}{i!\,j!\,(n-i-j)!}.$$

Using

$$(1 - u_1 - v_1)^{n-i-j} = \sum_{k+r<n-i-j}(u_2 - u_1)^k(v_2 - v_1)^r(1 - u_2 - v_2)^{n-i-j-k-r}$$
$$\cdot\frac{(n-i-j)!}{k!\,r!\,(n-i-j-k-r)!},$$

we get

$$B^n(f, P_1) = \sum_{i+j+k+r<n} f(i/n, j/n)\frac{n!}{i!\,j!\,k!\,r!\,(n-i-j-k-r)!}$$
$$\cdot u_1^i v_1^j(u_2 - u_1)^k(v_2 - v_1)^r(1 - u_2 - v_2)^{n-i-j-k-r} \qquad (8)$$

Since $f(P) \in \mathrm{Lip}_A\lambda$ and formulas (3), (4), (7), (8), we know that

$$|B^n(f, P_2) - B^n(f, P_1)| < A \sum_{k+r<n} (kh_2/n + rh_1/n)^\lambda \frac{n!}{k!\,r!\,(n-k-r)!}$$
$$\cdot (u_2-u_1)^k (v_2-v_1)^r (1-(u_2-u_1)-(v_2-v_1))^{n-k-r}$$
$$= A\cdot B^n((uh_2+vh_1)^\lambda; u_2-u_1, v_2-v_1)$$
$$< A((u_2-u_1)h_2 + (v_2-v_1)h_1)^\lambda$$
$$< A(\sqrt{2}\,\|P_2-P_1\|^\lambda) = \sqrt{2}^\lambda A \|P_2-P_1\|^\lambda. \quad (9)$$

Therefore the theorem is proved for the case 1. For the case of $u_2<u_1, v_2<v_1$, the similar argument shows that the theorem is also valid.

Case 2. Suppose $u_2>u_1, v_2<v_1$.

In this case, either $w_2>w_1$ or $w_2<w_1$, without loss of generality, we may suppose $w_2>w_1$. Because of symmetry, we can think of $B^n(f, P)$ as functions of variables u and w, then this case is reduced to the case 1, we still have

$$|B^n(f, P_2) - B^n(f, P_1)| < \sqrt{2}^\lambda A \|P_2-P_1\|^\lambda \cdot \quad (10)$$

For the case of $u_2<u_1, v_2>v_1$ (10) is also valid.

Summerizing the case 1 and 2, the theorem is confirmed.

At last, we would like to make some remarks:

1. The constant $\sqrt{2}^\lambda A$ is the best in a sense. Let

$$f(P) = f(x, y) = (x^2+y^2)^{\lambda/2}.$$

It is easy to see that

$$|f(P_2) - f(P_1)| < \|P_2-P_1\|^\lambda,$$

that is $f(P) \in \mathrm{Lip}_A\lambda$ and $A=1$. For $n=1$,

$$B^1(f, x, y) = x+y.$$

Take $(x, y) = (1/2, 1/2)$, since $\|(1/2\ 1/2) - (0,0)\| = 1/\sqrt{2}$, therefore

$$|B^1(f, 1/2, 1/2) - B^1(f, 0, 0)| = 1 = \sqrt{2}^\lambda \|(1/2, 1/2) - (0,0)\|^\lambda,$$

that is the inequlity in the theorem becomes equlity.

2. If T is an arbitrary acute or rectangular triangle, then the theorem is still valid.

3. If T is an arbitrary obtuse triangle, there is no constant C such that for all n and any T,

$$B^n(f, p) \in \mathrm{Lip}_C\lambda$$

holds. The reason is that the formula (3) may not hold in this case. A counterexample is as following.

Let T be a triangle with $(0,0)$, $(2m,0)$, $(m,1)$ as its vertices and $f(P)$ be as above.

Take $P_1=(m,1)$, $P_2=(m,0)$, in this case $\|P_2-P_1\|=1$ and $B^1(f, P_1) = (1+m^2)^{\lambda/2}$, $B^1(f, P_2) = 2^\lambda m^\lambda/2$, therefore

$$|B^1(f, P_1) - B^1(f, P_2)| = (1+m^2)^{\lambda/2} - 2^\lambda m^\lambda/2 > m^\lambda(1-2^{\lambda-1}) \xrightarrow[m\to\infty]{} \infty.$$

Reference

[1] Brown, B.M., Elliott, D.and Paget, D.F., J.of Appro.Theory, Vol.49(1987), 196—199.

[2] Chang, G.Z.and Davis, P.J., J.of Appro. Theory, Vol.40(1984) 11—28.

三角域上Bernstein多项式的Lipschitz常数

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摘要 设T是平面上以T_1, T_2, T_3为顶点的三角形，$f(p)$为定义在T上的函数，称

$$B^n(f,P):=\sum_{i+j+k=n} f(\frac{i}{n},\frac{j}{n},\frac{k}{n})B^n_{i,j,k}(P)$$

为f的n次Bernstein多项式，这儿

$$B^n_{i,j,k}(P):=\frac{n!}{i!j!k!}u^i v^j \omega^k$$

是Bernstein基函数，(u,v,w)是P关于T的重心坐标.

B.M.Brown等人对单变量的Benrstein多项式证明了如果$f\epsilon \mathrm{Lip}_A\lambda$，$0<\lambda\leqslant 1$，则对所有的$n$，都有$B^a(f,x)\epsilon \mathrm{Lip}_A\lambda$. 本文的目的是对定义在三角域$T:=\{(x,y):x\geqslant 0, y\geqslant 0, x+y\leqslant 1\}$上的Bernstein多项式证明了类似的结果：

设$f(P)\epsilon \mathrm{Lip}_A\lambda$，$0<\lambda\leqslant 1$，则对所有的$n$，$B^n(f,P)\epsilon \mathrm{Lip}_{\sqrt{2}^\lambda A}\lambda$，并且，在一定意义上，常数$\sqrt{2}^\lambda A$是最好的.

上述结果对于任意的锐角或直角三角形T，也是成立的.

最后还指出，当T可为钝角三角形时，则不存在同一常数C，使对所有的n和任意三角形T，有$B^n(f,P)\epsilon \mathrm{Lip}_C\lambda$.