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On the Asymptotics of the Cesaro Kernel on Unitary Groups*

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I. Introduction

Let U_n be the unitary group of order n. For $u \in L^1(U_n)$ the Fourier series, of u is (see [1])

$$u(U) \sim \sum_f N(f)\operatorname{tr}(c_f A_f(U)),\ U \in U_n$$

Here $N(f)$ is the order of the single-valued irreducible unitary representation $A_f(U)$ of U_n which takes $f=(f_1,\cdots,f_n)$ as its label ($f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n$ all are integers), and $C_f = \frac{1}{\omega_n}\int_{U_n} u(U)A_f(U')\dot{U}$. When $n=1$ this is the classical Fourier series on T, 1-demensional torus.

Gong Sheng [1] defined a variety of summation of Fourier series on unitary groups and a series of results of the classical Fourier analysis were extended to unitary groups. For example, he defined two types of Gesaro (C, a) summations of Fourier series on U_n, the type I and type II. Each of them is the classical Cesaro (C, a) summation of Fourier series when $n=1$.

Let $u \in L^1(U_n)$ and then the Cesaro (C, a) means of type II of its Fourier series is

$$\sigma_N^a(u,U) = \sum_{N > l_1 > \cdots > l_n > -N} A_{l_1}^a \cdots A_{l_n}^a N(f)\operatorname{tr}(C_f A_f(U)), \qquad (1)$$

where $A_k^a = \Gamma(a+N-|k|+1)\Gamma(N+1)/\Gamma(a+N+1)\Gamma(N-|k|+1)$, $l_j = f_j - j + n$, $j=1, \cdots, n$, and $a > -1$. The kernel corresponding to summation (1) is

$$K_N^a(U) = (-i)^{n(n-1)/2}((n-1)!\cdots 2!1!D(e^{i\theta_1},\cdots,e^{i\theta_n}))^{-1}$$
$$\times D\left(\frac{\partial}{\partial\theta_1},\cdots,\frac{\partial}{\partial\theta_n}\right)(k_N^a(\theta_1)\cdots k_N^a(\theta_n)),$$

where $e^{i\theta_1},\cdots, e^{i\theta_n}$ are the eigenvalues of U, $k_N^a(t)$ is the one-dimensional Cesaro kernel and

$$D(x_1,\cdots,x_n) = \prod_{1 < i < j < n}(x_i - x_j).$$

We call k_N^a the Cesaro-Gong kernel of type II on U_n. Therefore $\sigma_N^a(u,U) = = k_N^a * u(U)$.

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In the present paper we give the sharp estimates for Lebesgue constants $\|k_N^a\|_{L^1(U_n)}$ and an application of the results to the best conditions for the norm convergence of (C,a) means on U_n which extend and improve the results in [1] and [2].

2. Main Results

Denote by ρ_N^a the norm of K_N^a in $L^1(U_n)$, $\rho_N^a = \|K_N^a\|_{L^1(U_n)}$. Set

$$J_{n,a} = \begin{cases} n(n-1)/2-(n-2[a]-1)a-[a]([a]+1), & a \geq 0, \\ n(n-1)/2-na, & -1 < a < 0, \end{cases}$$

$$J_n = \begin{cases} n(n-2)/4, \text{ if } n \text{ even}, \\ (n-1)^2/4, \text{ if } n \text{ odd}, \end{cases}$$

where $[x]$ is the largest integer $\leq x$.

Our main results are as follows.

Theorem 1 (a) Suppose a is a nonnegative integer. Then

$$\rho_N^a = \begin{cases} C_1 N^{J_{n,a}} \operatorname{Log} N + O(N^{J_{n,a}}), \text{ if } a \leq (n-1)/2, \\ C_2 N^{J_n} \log N + O(N^{J_n}), \text{ if } a > (n-1)/2 \text{ and } n \text{ even}, \\ C_3 N^{J_n} + O(N^{J_n - 1}(\log N)^3), \text{ if } a > (n-1)/2 \text{ and } n \text{ odd}; \end{cases}$$

(b) Suppose $-1 < a < 0$, then $\rho_N^a = C_4 N^{J_{n,a}} + O(N^{J_{n,a}+a} \log N)$, where C_i $(i=1, 1,2,3,4)$ are the positive constants that can be determined explicitly.

In particular, when $a=0$ we obtain the asymptotics of the Dirichlet kernel D_N on U_n:

$$\|D_N\|_{L^1(U_n)} = CN^{n(n-1)/2} \log N + O(N^{n(n-1)/2})$$

Where $D_N(U) = \sum_{N > l_1 > \cdots > l_n > -N} N(f)\chi_f(U)$ and χ_f is the character of A_f.

Theorem 2 Let a be not an integer and $a > 0$. Then there exist two positive constants K_1 and K_2 such that

$$K_1 N^{J_{n,a}} \leq \rho_N^a \leq K_2 N^{J_{n,a}}, \text{if } a < [(n+1)/2],$$

$$K_2 N^{J_n} (\operatorname{lon} N)^{\varepsilon_n} \leq \rho_N^a \leq K_2 N^{J_n} (\log N)^{\varepsilon_n}; \text{if } a > [(n+1)/2].$$

3. The proofs of theorems

In order to prove our results, we need the following lemmas.

Lemma 1 Let k_N^a be the 1-dimensional Cessaro kernel and m the nonnegative integer.

(a) Suppose r is a positive integer and $r-1 < a < r$. Then for $t \in (N^{-1}, 2\pi)$ we have

$$\frac{d^m}{dt^m} k_N^a(t) = \sum_{k=0}^{m} N^{m-k-a} (2\sin(t/2))^{-a-1-k} \{ \sum_{j=0}^{k} a_{kj}^{(1)} (\cos t/2)^j$$

$$\times \sin((N + \frac{a+k-j+1}{2})t + \frac{(m-k-a)\pi}{2})\} + \sum_{k=0}^{m} \sum_{j=1}^{r-1} a_{kj}^{(2)} N^{-j} (2\sin(t/2))^{-j-k-1}$$

$$\times (\cos(t/2))^k \sin(\frac{j-1}{2}t + \frac{m-j-k}{2}\pi) + O(N^m (Nt)^{-a-1}).$$

(b) Suppose a is a nonnegative integer, $a = r-1$. Then for $t \in (N^{-1}, 2\pi)$ we have

$$\frac{d^m}{dt^m} k_N^a(t) = \sum_{k=0}^{m} b_k N^{m-k-a} (2\sin\frac{t}{2})^{-r-k} (\cos\frac{t}{2})^k \sin((N+\frac{r}{2})t + \frac{m-k-a}{2}\pi)$$

$$+ \sum_{k=0}^{m} \sum_{j=1}^{r-1} b_{kj} N^{-j} (2\sin(t/2))^{-j-k-1} + O(N^m (Nt)^{-\min(r,m+2)})$$

(c) For $-1 < a < 0$ and $t \in (N^{-1}, 2\pi)$ we have

$$\frac{d^m}{dt^m} k_N^a(t) = \Gamma(a+1) N^{m-a} \sin(Nt + \frac{m-1}{2}\pi) + \sum_{k=0}^{m} c_k N^{m-k-a}$$

$$\times (2\sin\frac{t}{2})^{-a-k-1} \sin((N + \frac{a+k+1}{2})t + \frac{m-k-a}{2}\pi) + O(N^{m+1}(Nt)^{-1})$$

Here $a_{ij}^{(1)}$, $a_{ij}^{(2)}$, b_{ij}, b_k, c_l are the constants which only detend on m and a.

Lemma 2 Let m be a nonnegative integer and set

$$\varphi_{s,a}(t) = \begin{cases} (-1)^m \sum_{k=0}^{\infty} \sum_{j=1}^{2m+2k} \frac{(-1)^{k+j}}{(j+a+1)(2k)!} \binom{2m+2k}{j} t^{2k}, & s = 2m, \\ (-1)^{m+1} \sum_{k=0}^{\infty} \sum_{j=0}^{2m+2k+2} \frac{(-1)^{k+j}}{(j+a+1)(2k+1)!} \binom{2m+2k+2}{j} t^{2k+1}, & s = 2m+1. \end{cases}$$

Then for $a > -1$ we have

$$\frac{d^s}{dt^s} k_N^a(t) = \varphi_{s,a}(Nt) N^{s+1} + Q(N^s + N^{s-a}).$$

Let m be an integer and $1 \leqslant m \leqslant n+1$. Set

$$D_N^{(m)} = \{\theta = (\theta_1, \cdots, \theta_n);\ \pi \geqslant \theta_n \geqslant \cdots \geqslant \theta_m \geqslant N^{-1} \geqslant \theta_{m-1} \geqslant \cdots \geqslant \theta_1 \geqslant 0\}.$$

Let $(s) = (s_1, \cdots, s_n)$ be a permutation of $1, 2, \cdots, n$ such that $s_{i_1} < \cdots < s_{i_{r-1}}$, $s_{i_r} < \cdots < s_{i_n}$, and $s_{i_k} = k$ for $k = 1, 2, \cdots, m-1$. Here $\{i_1, \cdots, i_{r-1}\} = \{1, 2, \cdots, r-1\}$ and r is a positive integer. Set

$$I_N = N^{r-1} \int_{D_N^{(m)}} N^{-j_r - \cdots - j_n} \prod_{k=1}^{n} (N\theta_{i_k})^{j_k} d\theta$$

where $i_k = k$ and s_{i_k} for $k = 1, 2, \cdots, m-1$ and $k = m, m+1, \cdots, n$ respectively, and j $j_k (k = 1, 2, \cdots, n)$ are the integers such that $j_k \in \{0, 1, \cdots, n-1\}$ (for $k = 1, \cdots, m-1$) $j_k < -1$ (for $k = m, m+1, \cdots, r-1$), $j_k \geqslant -1$ (for $k = r, r+1, \cdots, n$), and there exists only one $\mu \in \{r, r+1, \cdots, n\}$ satisfying $j_\mu = -1$.

Lemma 3 If $s_{i_k} = k (k = 1, 2, \cdots, n)$ and $j_r = -1$, then

$$I_N = N^{r - j_r - \cdots - j_n - 1} \int_{D_N^{(m)}} \prod_{k=1}^{n} (N\theta_k)^{j_k} d\theta,$$

otherwise $I_N = O(1)$.

We omit the proofs of Lemma 1 through Lemma 3 because of limited space.

Now we turn to the proofs of theorems.

Proof of Theorem 1 Let $c_n=(D(n-1,\cdots,1,0)(2\pi)^n)^{-1}$ and $D(e^{i\theta})=D(e^{i\theta_1},\cdots,e^{i\theta_n})$. Set

$$\widetilde{K}_N^a(\theta)=D(\frac{\partial}{\partial\theta_1},\cdots,\frac{\partial}{\partial\theta_n})\prod_{j=1}^n k_N^a(\theta_j).$$

and $D=\{\theta=(\theta_1,\cdots\ \theta_n);\pi\geqslant\theta_n\geqslant\cdots\geqslant\theta_1\geqslant 0\}$, $D_{i_1\cdots i_k}=\{\theta=(\theta_1,\cdots,\theta_n);0\leqslant\theta_n\leqslant\pi,$ $a_{j+1}\leqslant\theta_j\leqslant\beta_{j+1}, j=1,2,\cdots,n-1\}$, where

$$a_j=\begin{cases}0, & j\neq i_p,\\ -|\theta_j|, & j=i_p,\end{cases}\qquad \beta_j=\begin{cases}0, & j=i_p,\\ |\theta_j|, & j\neq i_p,\ p=1,2,\cdots,k.\end{cases}$$

Then we have

$$\rho_N^a=c_n\int_{|\theta_1|\leqslant\cdots\leqslant|\theta_n|\leqslant\pi}|\widetilde{K}_N^a(\theta)D(e^{i\theta})\mathrm{d}\theta$$

$$=2c_n(I_0+\sum_{k=1}^{n-1}\sum_{\substack{i_1,\cdots,i_k=2\\ i_1>\cdots>i_k}}^{n}I_{i_1\cdots i_k})\tag{2}$$

Where

$$I_0=\int_D|\widetilde{K}_N^a(\theta)D(e^{i\theta})|\mathrm{d}\theta,\quad I_{i_1\cdots i_k}=\int_{D_{i_1\cdots i_k}}|\widetilde{K}_N^a(\theta)D(e^{i\theta})|\mathrm{d}\theta.$$

Let $\widetilde{\theta}_j=-\theta_j$ for $j=i_p-1$ and $\widetilde{\theta}_j=\theta_j$ for other $j(p=1,2,\cdots,k)$. Then for $\theta\in D$ we have

$$|D(e^{i\theta})|=\sum_{(j)}\delta_{j_1\cdots j_n}^{0\cdots n-1}(-1)^{\tau(i_1,\cdots,i_k)}\prod_{s=1}^n(\sin\frac{\theta_s}{2})^{j_s}(\cos\frac{\theta_s}{2})^{n-1-j_s}$$

where $(j)=(j_1,\cdots,j_n)$ is a permutation of $0,1,\cdots,n-1$, $\delta_{j_1\cdots j_n}^{0\cdots n-1}$is the symbol of (j),

$$\delta_{j_1\cdots j_n}^{0\cdots n-1}=\begin{cases}1, & \text{if } (j)\text{ even},\\ -1, & \text{if } (j)\text{ odd},\end{cases}$$

and $\tau(i_1,\cdots,i_k)=i_1+\cdots+i_k+j_{i_1}+\cdots+j_{i_k}-2k$. Therefore we obtain

$$I_{i_1\cdots i_k}=\sum_{(j)}\delta_{j_1\cdots j_n}^{0\cdots n-1}(-1)^{\tau(i_1,\cdots,i_k)}\sum_{m=1}^{n+1}I_{i_1\cdots i_k}^{j_1\cdots j_n}(m),\tag{3}$$

where

$$I_{i_1\cdots i_k}^{j_1\cdots j_n}(m)=\int_{D_N^{(m)}}|\widetilde{K}_N^a(\widetilde{\theta})|\prod_{s=1}^n(\sin\frac{\theta_s}{2})^{j_s}(\cos\frac{\theta_s}{2})^{n-1-j_s}\mathrm{d}\theta$$

and $D_N^{(m)}=\{\theta\in D;\theta_m\geqslant N^{-1}\geqslant\theta_{m-1}\}$. Now we fix $k=k_0$ and set $J_N=I_{i_1\cdots i_{k_0}}^{j_1\cdots j_n}(m)$.

Let $a=r-1$ (r is a positive integer) and $E_l=A_l\cup B_l$ for an integer $l\geqslant 0$, where $A_l=\{r+k;k=0,1,\cdots,l\}$ and $B_l=\{j+k+1;j=1,\cdots,r-1,k=0,1,\cdots,l\}$. For $s\in E_l$ set

$$f_s^l(\theta_j)=\begin{cases}h_s^l(\theta_j), & \text{if } j\neq i_p-1,\\ (-1)^l h_s^l(\theta_j), & \text{if } j=i_p-1,\ p=1,\cdots,k_0,\end{cases}$$

$$g_l(\theta_j)=\begin{cases}\varphi_{l,r-1}(\theta_j), \text{ if } l \text{ even},\\ \Psi_{l-1,r-1}(\theta_j), \text{ if } l \text{ odd},\end{cases}$$

where

$$h_s^l(t)=\begin{cases}\sin((N+r/2)t+(l-s+1)\pi/2)(\cos(t/2))^k, \text{ if } s=r+k\in A_l,\\ \sin((j-1)t/2+(1-s+1)\pi/2)(\cos(t/2))^k, \text{ if } s=j+k+1\in B_l,\end{cases}$$

$$\psi_{l,r-1}(\theta_j)=\begin{cases}\varphi_{l,r-1}(\theta_j), \text{ if } j\neq i_p-1,\\ -\varphi_{l,r-1}(\theta_j), \text{ if } j=i_p-1,\ p=1,2,\cdots,k_0.\end{cases}$$

By Lemma 1 and Lemma 2 we have

$$\begin{aligned}J_N=J_N(j_1,\cdots,j_n,m)&=N^{n(n+1)/2}\int_{D_N^{(m)}}\Big|\sum_{(l)}\delta^{0\cdots n-1}_{l_1\cdots l_n}\prod_{i=1}^{m-1}g_{l_i}(N\theta_i)\theta_i^{j_i}\\ &\times\prod_{l=m}^{n}\sum_{s\in E_{l_i}}a_sN^{-s-s'}f_s^{l_i}(\theta_i)\Big(2\sin\frac{\theta_i}{2}\Big)^{-s+j_i}\Big(\cos\frac{\theta_j}{2}\Big)^{n-1-j_i}\Big|d\theta\\ &+O(N^{J_{n,*}-1}\log N+N^{J_*-1}\log N),\end{aligned}\tag{4}$$

where $s'=i-k$ if $s=i+k+1\in B_l$ for some $i\in\{0,1,\cdots,n-1\}$ otherwise $s'=0$.

Suppose $r\leqslant(n+1)/2$ and choose $m\leqslant r$. Let $(\mu)=(\mu_1,\cdots,\mu_{r-m})$ denote the permutation of $m, m+1,\cdots,r-1$ and $(v)=(v_1,\cdots,v_{n-r})$ the permutation of $r+1,\cdots,n$. Denote by E_{mqi} the set of all permutations (j), (l), (μ), (v) such that $\{j_1,\cdots,j_{r-1}\}=\{0,1,\cdots,r-2\}$, $j_r=r-1$, $l_{\mu_s}-j_{\mu_s}>-1$ (for $s=1,2,\cdots,n-q-r-m+2$), $l_{v_s}-j_{v_s}<-1$ (for $s=1,2,\cdots,i$), and $\{l_{v_1},\cdots,l_{v_i}\}=\{0,1,\cdots,i-1\}$, $l_r=r-2$ when $i=r-2$, where $i=r-2,r-1$ and $q=n-2r+2,\cdots,n-r-m+2$. In addition, for (l), (j), (μ), $(v)\in E_{mqi}$ we set

$$\begin{aligned}S_{(\mu)(l)qi}(t_1,\cdots,t_r)&=\psi_{il_r}(t_r)\prod_{\lambda=1}^{m}g_{l_\lambda}(t_\lambda)t_\lambda^{j_\lambda}\\ &\times\prod_{\lambda=1}^{n-q-r-m+2}\sum_{\sigma_\lambda=1}^{r-1}a^{(1)}_{\sigma_\lambda}((l),(\mu),q)t_{\mu_\lambda}^{s_1(\mu_\lambda)}\prod_{\lambda=n-q-r-m+3}^{r-m}\sum_{\tau_\lambda=0}^{l_{\mu_\lambda}}a^{(2)}_{\tau_\lambda}((l),(\mu),q)\\ &\times\sin(t_{\mu_\lambda}+(l_{\mu_\lambda}-\tau_\lambda-r+1)\pi/2)t_{\mu_\lambda}^{s_2(\mu_\lambda)}\end{aligned}$$

$$\begin{aligned}\eta_{(v)(l)qi}(y_{r+1},\cdots,y_n;z_{r+1},\cdots,z_n)&==\prod_{\lambda=1}^{i}\sum_{\tau_\lambda=0}^{l_{\mu_\lambda}}a_{\tau_\lambda}((l),(v),q)\\ &\times\Big(\cos\frac{z_{v_\lambda}}{4}\Big)^{n-3-s_3(v_\lambda)}\Big(2\sin\frac{z_{v_\lambda}}{4}\Big)^{s_3(v_\lambda)}\prod_{\lambda=i+1}^{n-r}a^{(4)}_\lambda((l),(v),q)\\ &\times\Big(\cos\frac{z_{v_\lambda}}{4}\Big)^{n-1-r-s_4(v_\lambda)}\Big(2\sin\frac{z_{v_\lambda}}{4}\Big)^{s_4(v_\lambda)}\sin\Big(y_{v_\lambda}+\frac{\pi}{2}(l_{v_\lambda}-r+1)+\frac{rz_{v_\lambda}}{4}\Big),\end{aligned}$$

where $s_1(\mu_\lambda)=j_{\mu_\lambda}-\sigma_\lambda-l_{\mu_\lambda}-1$, $s_2(\mu_\lambda)=j_{\mu_\lambda}-\tau_\lambda-r$, $s_3(v_\lambda)=j_{v_\lambda}-\tau_\lambda-2$, $s_4(v_\lambda)=j_{v_\lambda}-r$, and

$$\psi_{i,l_r}(t)=\begin{cases}h_{i_1\cdots i_{k_0}}(r)\sin\big(t+\frac{\pi}{2}(l_r-r+1)\big), \text{ if } i=r-1,\ l_r\geqslant r-1,\\ h_{i_1\cdots i_{k_0}}(r), \text{ if } i=l_r=r-2,\\ 0, \text{ for other } i, l_r,\end{cases}$$

$$h_{i_1\cdots i_{k_0}}(s)=\begin{cases}1, \text{ if } s\neq i_p-1,\\ (-1)^{l_s}, \text{ if } s=i_p-1;\ p=1,2,\cdots,k_0.\end{cases}$$

Now replace θ_k by $t_k=N\theta_k$ for $k=1,2,\cdots,r$. By Lemma 3 and (4) we can show that

$$J_N=N^{J_{n,a}}\int_{D_N}\zeta(t_1,\cdots,t_r; N\theta_{r+1},\cdots,N\theta_n; 2\theta_{r+1},\cdots,2\theta_n)$$
$$t_r^{-1}\mathrm{d}t_1\cdots\mathrm{d}t_r\mathrm{d}\theta_{r+1}\cdots\mathrm{d}\theta_n+O(N^{J_{n,a}}), \qquad (5)$$

Here $D_N=\{(t_1,\cdots,t_r,\theta_{r+1},\cdots,\theta_n);\ 0\leqslant t_1\leqslant\cdots\leqslant t_{m-1}\leqslant 1\leqslant t_m\leqslant\cdots\leqslant t_{r-1},\ 1\leqslant t_r\leqslant N\pi, N^{-1}\leqslant \leqslant\theta_{r+1}\leqslant\cdots\leqslant\theta_n\leqslant\pi\}$, χ_E is the characteristic function of the set E, and

$$\zeta(x_1,\cdots,x_r; y_{r+1},\cdots,y_n; z_{r+1},\cdots,z_n)=$$
$$=\Big|\sum_{(l),\ (\mu),\ (\nu)}\delta^{0\cdots n-1}_{l_1\cdots l_n}\sum_{q=n-2r+2}^{n-r-m+2}\sum_{i=r-2}^{r-1}\chi_{E_{mqi}}((j),\ (l),\ (\mu),\ (\nu))$$
$$\xi_{(\mu)(l)qi}(x_1,\cdots,x_r)\eta_{(\nu)(l)qi}(y_{r+1},\cdots,y_n; z_{r+1},\cdots,z_n).$$

Denote by $t_r^{-1}\varphi_N(t_1,\cdots,t_r,\theta_{r+1},\cdots,\theta_n)$ the integrand of the integral in (5) and set

$$\psi_N(t_r)=\int_{D_r\cup D'_N}\varphi_n(t_1,\cdots,t_r,\theta_{r+1}\cdots,\theta_n)\mathrm{d}t_1\cdots\mathrm{d}t_{r-1}\mathrm{d}\theta_{r+1}\cdots\mathrm{d}\theta_n$$

where $D_r=\{(t_1,\cdots,t_{r-1});\ 0\leqslant t_1\leqslant\cdots\leqslant t_{m-1}\leqslant 1\leqslant t_m\leqslant\cdots\leqslant t_{r-1}\}$ and $D'_N=\{(\theta_{r+1},\cdots,\theta_n);\ N^{-1}\leqslant\theta_{r+1}\leqslant\cdots\leqslant\theta_n\leqslant\pi\}$. Then ψ_N is a $2\pi-$periodic continuous function on $[0,+\infty)$. Therefore we obtain

$$J_N=\Big(\int_0^{2\pi}\psi_N(t)\mathrm{d}t\Big)N^{J_{n,a}}\log N+O(N^{J_{n,a}}). \qquad (6)$$

Set

$$F_N(t_1,\cdots,t_r)=\int_{D'_N}\varphi_N(t_1,\cdots,t_r,\theta_{r+1},\cdots,\theta_n)\mathrm{d}\theta_{r+1}\cdots\mathrm{d}\theta_n. \qquad (7)$$

Then

$$\psi_N(t)=\int_{D_r}F_N(t_1,\cdots,t_{r-1},t)\mathrm{d}t_1\cdots\mathrm{d}t_{r-1},\ t\in[0,2\pi]. \qquad (8)$$

In the integral expression of F_N we replace $N\theta_j$ by $x_j (j=r+1,\cdots,n)$ and we obtain

$$F_N(t_1,\cdots,t_r)=2^{r-n}\int_0^{2\pi}\cdots\int_0^{2\pi}\widetilde{\varphi}_N(t_1,\cdots,t_r,y_{r+1},\cdots,y_n)$$
$$+O(N^{-1}\xi(t_1,\cdots,t_r)),$$

where

$$\widetilde{\varphi}_N(t_1,\cdots,t_r,y_{r+1},\cdots,y_n)=$$
$$=\Big[\frac{N}{2}\Big]^{r-n}\sum_{i_n=1}^{[N/2]-1}\sum_{i_{n-1}=1}^{i_n-1}\cdots\sum_{i_{r+1}=1}^{i_{r+2}-1}\zeta(t_1,\cdots,t_r; y_{r+1},\cdots,y_n; a_N(y_{r+1},i_{r+1}),\cdots,a_N(y_n,i_n))$$

$$\xi(t_1,\cdots,t_r)=\sum_{(l),\ (\mu)}\sum_{q=n-2r+2}^{n-r-m+2}\sum_{i=r-2}^{r-1}|\xi_{(\mu)(l)qi}(t_1,\cdots,t_r)|$$

and $a_N(y_s,i_s)=[N/2]^{-1}(y_s+2\pi i_s)\ (s=r+1,\cdots,n)$.

In the other hand, we take the following function

$$\widetilde{\varphi}(t_1,\cdots,t_r;y_{r+1},\cdots,y_n)=$$
$$=(2\pi)^{r-n}\int\cdots\int_{2\pi\geqslant z_n\geqslant\cdots\geqslant z_{r+1}\geqslant 0}\xi(t_1,\cdots,t_r;y_{r+1},\cdots y_n;z_{r+1},\cdots,z_n)\mathrm{d}z_{r+1}\cdots\mathrm{d}z_n.$$

Then there exist $\beta_N(i_s)\in[[N/2]^{-1}2\pi i_s,\ [N/2]^{-1}2\pi(i_s+1)]\ (s=r+1,\cdots,n)$ such that

$$\widetilde{\varphi}(t_1,\cdots,t_r;y_{r+1},\cdots,y_n)=$$
$$=[\frac{N}{2}]^{r-n}\sum_{i_n=1}^{[N/2]}\sum_{i_{n-1}=1}^{i_n-1}\cdots\sum_{i_{r+1}=1}^{i_{r+2}-1}\xi(t_1,\cdots,t_r;y_{r+1},\cdots,y_n;\beta_N(i_{r+1}),\cdots,\beta_N(i_n))$$
$$+O(N^{-1}\xi(t_1,\cdots,t_r)).$$

Therefore we have

$$\widetilde{\varphi}_N(t_1,\cdots,t_r;y_{r+1},\cdots,y_n)-\widetilde{\varphi}(t_1,\cdots,t_r;y_{r+1},\cdots,y_n)=O(N^{-1}\xi(t_1,\cdots,t_r)),\quad (9)$$

and for $1\leqslant m\leqslant r,\ j_r=r-1,\ \{j_1,\cdots,j_{r-1}\}=\{0,1,\cdots,r-2\}$,

$$J_N=I^{j_1\cdots j_n}_{i_1\cdots i_{k_0}}(m)=C_{m,k_0}N^{J_{n,a}}\log N+O(N^{J_{n,a}}),$$

here

$$C_{m,k_0}=\int_{\Delta_n^{(m)}}\widetilde{\varphi}(t_1,\cdots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n$$

and $\Delta_n^{(m)}=\{(t_1,\cdots,t_n);\ 0\leqslant t_1\leqslant\cdots\leqslant t_{m-1}\leqslant 1\leqslant t_m\leqslant\cdots\leqslant t_{r-1},\ 0\leqslant t_r,\cdots,t_n\leqslant 2\pi\}$. Thus we complete the proof of assertion (a) for $a\leqslant(n-1)/2$ from (2), (3), and (6)—(9). The rest of Theorem 1 can be proved similarly.

Proof of Theorem 2 From (2) we can see

$$\rho_N^a\geqslant\int_{D_N^{(m)}}|\widetilde{K}_N^a(\theta)D(e^{i\theta})|\mathrm{d}\theta,\ m=1,2,\cdots,n+1.$$

Suppose $a<[(n+1)/2]$, then choose $m=[a]+2$. By the argument similar to the proof of Theorem 1 we can obtain

$$\int_{D_N^{(m)}}|\widetilde{K}_N^a(\theta)D(e^{i\theta})|\mathrm{d}\theta=CN^{J_{n,a}}+O(N^{J_{n,a}-1}\log N)$$

Therefore there exists a positive constant A such that $\rho_N^a\geqslant AN^{J_{n,a}}$. This completes the proof of the estimate from below for $a<[(n+1)/2]$. For $a>[(n+1)/2]$ choose $m=[(n+1)/2]+1$. Then the same argument gives the estimate from below for $a>[(n+1)/2]$. The upper estimates in Theorem 2 follow from our lemmas. The proof of Theorem 2 is completed.

4. An Application

The sharp estimates of the previous theorems allow us to obtain the best conditions for the norm convergence of the Cesaro means of type II in the space $E=C(U_n)$ or $E=L^1(U_n)$.

Let u_n be the Lie algebra of U_n. For $u\in E$ and $X\in u_n$ we define

$$Xu(V)=\lim_{t\to 0}\frac{u(V\exp tX)-u(V)}{t},\ V\in U_n$$

as the Lie derivative of u in the sense of the norm of E. Set $E^{(0)}=E$, and

$$E^{(s)}=\{\varphi\in E;\ X\varphi\in E^{(s-1)},\ X\in u_n\}$$

for a positive integer s. let $\omega_k(t,u,E)$ denote the modulus of continuity of u in E, defined by

$$\omega_k(t,u,E)=\operatorname{Sup}\{\|\Delta_U^k u\|_E;\ d(I,U)\leqslant t\}$$

Here d denotes the geodesic distance on U_n and, for U, $V\in U_n$, and a positive integer k,

$$\Delta_U^k u(V)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}u(U^{-j}V)$$

Finally if $X_1,\cdots,X_{n^2}$ is a basis for u_n we define

$$\omega_k(t,u^{(s)},E)=\sum_{i_1,\cdots,i_s=1}^{n^2}\omega_k(t,X_{i_1}\cdots X_{i_s}u,E).$$

Applying the previous theorems and similar to [3] we can obtain the following result.

Proposition (a) Let $u\in E^{(s)}$. If either of the following conditions is satisfied,

(i) $[a]\leqslant(n-1)/2$ and $\omega_2(t,u^{(s)},E)=o(t^{J_{n,a}}|\log t|^{-\sigma_a})$,

(ii) $[a]>(n-1)/2$ and $\omega_2(t,u^{(s)},E)=o(t^{J_n}|\log t|^{-\varepsilon_n})$, $t\to0^+$,

then $\sigma_N^a(u)\to u$ in E as $N\to+\infty$. Here $\varepsilon_n=1$ if n even otherwise $\varepsilon_n=0$; $\sigma_a=1$ if a is a nonnegative integer otherwise $\sigma_a=0$.

(b) There exists a central function $u_a\in E^{(s)}$ for each $a>-1$, where s is the largest integer smaller than $\lambda_{n,a}$, such that $\omega_2(t,u_a^{(s)},E)=O(t^{\lambda_{n,a}}|\log t|^{-\varepsilon_{n,a}})$ as $t\to0$ but $\sigma_N^a(u_a)$ does not converge to u_a in E. Here $\lambda_{n,a}$ denotes $J_{n,a}$ and J_n, $\varepsilon_{n,a}$ denotes σ_a and ε_n, for $[a]\leqslant(n-1)/2$ and $[a]>(n-1)/2$ respectively.

References

[1] Gong Sheng, Harmonic analysis on classical groups, Science Press, Beijing, China, 1983.

[2] Zheng Xuean, Acta ScientiarumcNaturalium Universitatis Anhuiensis, No.1 2, 1982, 28~46.

[3] D.I. Cartwright and P.M. Soardi, J. Approx. Theory, 38(1983), 344, 353.

[4] A. Zygmund, Trigonometric Series, Vol.I, Ch.Ⅲ, Cambridge, 1959.

酉群上Fourier级数的Cesaro求和

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摘要 设u_n为n阶酉群。$u\in L^1(U_n)$的Fourier级数的第二型Cesáro平均为$\sigma_N^a(u,U)=K_N^a*u(U)$，其中

$$K_N^a(U)=\sum_{N\geqslant l_1>\cdots>l_n\geqslant -N}A_{l_1}^a\cdots A_{l_n}^a N(f)\chi_f(U),\ U\in U_n$$

为相应的核函数．本文给出“Lebesgue常数”$\|K_N^a\|_{L^1(U_n)}$的精确估计，并由此建立了酉群上函数的Fourier级数按第二型Cesáro求和收敛于自身的条件．