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Some Nonlinear Elliptic Problems with Linear Part at Resonance*

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Ⅰ. Introduction

Let $\Omega\subset R^n$ be a bounded regular domain, $\{\lambda_i\}_1^\infty$ be the sequence of the eigenvalues of the operator $-\Delta$ on $H_0^1(\Omega)$. We suppose always that $k\geqslant 1$ is fixed, φ is a eigenfunc ion corresponding to λ_k with $\int_\Omega\varphi^2=1$, and $h\in H^{-1}(\Omega)$ such that $\int_\Omega h\varphi=0$. Consider the following problem

$$(P_t)\qquad \begin{cases} -\Delta u-\lambda_k u+g(x,u)=t\varphi+h & \text{in } \Omega \\ u=0 & \text{on } \Omega \end{cases} \qquad (1.1)$$

In past several years, many results on this problem are obtained in the case of $k=1$ (see [3], [4]). In this paper, we consider the case of $k>1$ by use of the technique of connected set and the continuum theory for o-epi maps (see [6]), several existence and multiplicity results for (P_t) are esteblished when $k>1$ and λ_k is simple.

Our main results are as follows

Theorem Ⅰ Suppose

(g_1). $g(x,s)=g(s)$, $\forall x\in\Omega$ and $g:R\to R$ is a continuous periodic function with period T and periodic primitive.

(H_3). $k>1$ and λ_k is simple.

(H_4'). $\lambda_{k-1}<\lambda_k+g'(s)<\lambda_{k+1}$, $k>1$, $\text{const}<\lambda_1+g'(s)<\lambda_2$.

Then for every h there are two numbers $\tau_1,\tau_2:\tau_1<0<\tau_2$ such that

(i). (P_t) has a solution if and only if $t\in[\tau_1,\tau_2]$.

(ii). If $t\in(\tau_1,\tau_2)-\{0\}$, then (P_t) has at least two solutions.

Remark The same result was given in [1, theorem 2] for $k=1$. We extend this result to the case of $k>1$ with additional condition (H_4').

Theorem 2 Suppose

(H_2) $g:\Omega\times R\to R$ is measurable in x for every $s\in R$, and $g\in C^1$ in s a.e. on Ω.

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(H_3) $k>1$ and λ_k is simple.

(H_4') $\lambda_{k-1}\leqslant\lambda_k+g'(x,s)\leqslant\lambda_{k+1}$, $k>1$

const$\leqslant\lambda_1+g'(x,s)\leqslant\lambda_2$.

where $g'(x,s)=(\partial/\partial s)g(x,s)$.

(H_5) $\sup\{|g(x,s)|:(x,s)\in\Omega\times\mathbf{R}\}=d<+\infty$

If for every $x\in\Omega$,

$$\lim_{s\to\pm\infty} sg(x,s)=\mu>0.$$

Then for every $h\in H_0^1$, there are two numbers $\tau_1,\tau_2:\tau_1<0<\tau_2$ such that

(i) (P_t) has solutions if and only if $t\in[\tau_1,\tau_2]$.

(ii) If $t\in(\tau_1,\tau_2)-\{0\}$, then ($P_t$) has at least two distinct solutions.

Remark The same result was showed in [4, theorem 5.2] under the following asymptotic uniform condition

$$(H_4)\quad\begin{cases}\lambda_{k-1}<\text{const}\leqslant\lambda_k+g'(x,s)\leqslant\text{const}<\lambda_{k+1},\\ \text{const}\leqslant\lambda_1+g'(x,s)\leqslant\text{const}<\lambda_2\end{cases}$$

In our theorem, this condition was replaced by asymptotic non-uniformcondition.

Theorem 3 The condition $q<\nu(-\Delta-\lambda_k I)$ of [3, proposition 2.4] can be replaced by the condition $q\leqslant\nu(-\Delta-\lambda_k I)$, the same result is still ture.

2. Lyapunov-Schmidt Procedure

Denoted by (,) and $(,)_1$ the innerproducts in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, $\|\cdot\|$, $\|\cdot\|_1$ be the corresponding norms. Let us denoted by $L_k:H_0^1(\Omega)\to H_0^1(\Omega)$ the linear oprator defition by

$$(L_k u,v)_1=-\int\nabla u\nabla v+\lambda_k(u,v). \tag{2.1}$$

Then (1.1) is epuivalent to

$$L_k u+Gu=f, \tag{2.2}$$

where G and f are respectively defined by

$$(Gu,v)_1=-\int_\Omega g(x,u)v$$

and

$$(f,v)_1=-\int_\Omega(+\varphi+h)v.$$

Denoted by V the kernel of L_k and by $V^\perp$ its L^2-orthogonal complement. Let P and Q be the projections onto V and $V^\perp$ respectly Appling P and Q to (2.2), then (2.2) becomes

$$L_k w+QG(s\varphi+w)=Qf \tag{2.3}$$

$$PG(s\varphi+w)=Pf \tag{2.4}$$

where $w\in V^\perp$.

Let $k=(L_k|V^\perp)^{-1}:V^\perp\to V^\perp$ and $T_s=kQ[f-G(s\varphi+w)]$, then each s, T_s is compact. Denote by $S_h\subset R\times V^\perp$ the set of solutions of (2.3), i.e.

$$S_h = \{(s,w)\in R\times V^{\perp} | w\in H_0', L_{kw} + QG(s\varphi + w) = h\}.$$

By the uniform boundedness of g and Poincare's inquality [7, p68] it follows that T_s maps into the ball $\overline{B}_\rho = \{w\in V^{\perp} \mid \|w\| < \rho\}$, where

$$\rho = \|k\|(\|Qf\|_1 + \text{const}|\Omega|^{1/2}\sup|g(x,t)|). \qquad (*)$$

Therefore, by Schauder s fixed point theorem, $P_{\text{roj}_R} S_h = R$. Now, system (2.3), (2.4) equivalent to the following equation

$$\Phi(s,w) = t. \qquad (2.5)$$

in S_h, where the maping $\Phi: R\times V^{\perp}\to R$ is given by $\Phi(s,w) = (G(s\varphi + w), \varphi)_1$ i.e. (P_t) is equivalent to the equation $\int_\Omega g(x, s\varphi + w)\varphi dx = t$ in S_h.

3. Study of S_h

Definition Let E, F be Banach spaces, $U\subset E$ be a open and bounded set and $f: U\to F$ be continuous map such that $f(x) = 0$ for every $x\in\partial U$. We say that f is 0-epi if for every continous and compact map $h: U\to F$ such that $h(x) = 0$ for every $x\in\partial U$ the nonlinear oprator equation $f(x) = h(x)$ has a solution $x\in U$.

We recall first some results on the structure of the set of solution of the equation $f(x) = 0$ where f is 0-epi [6].

Proposition Let $f: U\to F$ be 0-epi and proper. Assume that for every $\varepsilon > 0$ and every $y\in f^{-1}(0)$, there exists a continuous and compact map $h_\varepsilon: U\to F$ such that

(i) $h_\varepsilon(y) = 0$;

(ii) $\|h_\varepsilon(x)\| < \varepsilon$ for all $x\in U$,

(iii) the set of solution of the equation $f(x) = h_\varepsilon(x)$ is ε-chained. Then $f^{-1}(0)$ is nonempty, connected and compact.

By using above theory, we study the structure of S_h now.

Lemma 3.1 Let $g: \Omega\times R\to R$ be bounded, λ_k is simple and let (H_4') hold. Then $S_h^{s_0} = \{(s_0, w) | (s_0, w)\in S_h\}$ is nonempty, connected and compact.

Proof Since $I - T_{s_0}$. is a completely continuous filed, we know that $I - T_{s_0}$ is proper and 0-epi (see the proof of [6, corollary 2.2]). Hence, in particular, $(I - T_{s_0})^{-1}(0)$ is bounded. Let $U\subset V^{\perp}$ be an open bounded set containing $(I - T_{s_0})^{-1}(0)$, let $\varepsilon > 0$ and let $y\in(I - T_{s_0})^{-1}(0)$. Construct an approximation $g_\varepsilon = (1 - \frac{\varepsilon}{3\rho})g$ and define

$$h_\varepsilon(w) = kQ[G(s_0\varphi + w) - G_\varepsilon(s_0\varphi + w)] + kQ[G_\varepsilon(s_0\varphi + y) - G(s_0\varphi + y)] \qquad (3.1)$$

where $G_\varepsilon: H_0^1\to H_0^1$, is defined by

$$(G_\varepsilon u, v)_1 = -\int_\Omega (1 - \frac{\varepsilon}{3\rho}) g(x,u) v,$$

for all $u\in H_0^1$. Then h_ε satisfies (i) and (ii) of proposition above. In order to check assumption (iii) of the proposition, we have only to show that the equation

$$(\mathrm{I}-T_{s_0})(w)=h_\varepsilon(w)\ . \tag{3.2}$$

has only one solution.

In fact, from (3.1), we know that (3.2) can be written as follows

$$w-kQ[f-G_\varepsilon(s_0\varphi+w)]=kQ[G_\varepsilon(s_0\varphi+y)-G(s_0\varphi+y)]. \tag{3.3}$$

(3.3) is equivalent to the equation

$$L_k w+QG_\varepsilon(s_0\varphi+w)=QG_\varepsilon(s_0\varphi+y)-QG(s_0\varphi+y)+Qf \tag{3.4}$$

we have only to show that the left satisfies all conditions of 4, lemma 2.2. From the definition of g_ε, we know that $(H_2)(H_3)$ (H_5) of [4, lemma 2.2] are satisfied. From (H_4') and the definition of g_ε it follows that (H_4) of [4, lemma 2.2] is also satisfied. Therefore equation (3.2) has only one solution. Clearly h_ε is compact. By Proposition, $QS_h^{s_0}=(\mathrm{I}-T_{s_0})^{-1}(0)$ is nonempty, connected and compact. So is $S_h^{s_0}$.

Lemma 3.2 Assume λ_k is simple, g is continuous and bounded, and (H_4') hold. Then $S_h\subset R^1\times\overline{B_\rho}$ is a connected set.

Proof This is a direct corollary of [2, theorem 0] and lemma 3.1.

Under the conditions of Theorem 3, let $g_\varepsilon=(1-\frac{\varepsilon}{2})g$, then Lipschitz constant of q_ε satisfies $q_\varepsilon<\nu(-\Delta-\lambda_k\mathrm{I})$. Arguing asimilarly, we obtain

Lemma 3.3 Assume λ_k is simple, g is bounded and continuous, and $q<\nu(-\Delta-\lambda_k\mathrm{I})$ hold. Then S_h is a connected set.

Remark In this case, by contraction mapping principle, the equation

$$(\mathrm{I}-T_{s_0})w=h_\varepsilon(w)\ .$$

has only one solution.

4. Proofs of Theorems

Proof of Theorem 1 Let $\tau=\{t\in\mathrm{R}\,|\,(P_t)$ has a solution$\}$, $\tau_1=\inf\tau$, $\tau_2=\sup\tau$, By [1, theorem 1], $0\in\tau$. Since g is bounded, the τ has to be bounded. This implies that

$$-\infty<\tau_1\leqslant 0\leqslant\tau_2<+\infty$$

and by [1, corollary 5, 11], (P_t) has solutions for $t=\tau_1\,\tau_2$, From (2.5) and Lemma 3.2, it follows τ is connected. Therefore all that we have to show that (P_t) has at least two solutions if $t\in(\tau_1,\tau_2)-\{0\}$.

Let $W=\{w\in V^\perp|\,(s,w)\in S_h\}$. By (*), W is bounded in H_0^1, hence W is precompact for the convergence in measure. By [9, P.53] $\nabla\varphi\neq 0$ a.e in Ω. Now [1, prosition 2.1] gives

$$\lim_{|s|\to\infty}\int g(s\varphi+w)\varphi=0$$

uniformly for $w\in W$. This is

$$\lim_{|s|\to\infty}\Phi(s,w)=0,$$

uniformly for $w \epsilon W$. In particular

$$\lim_{|s|\to\infty} \Phi(s, w_s) = 0 \qquad \forall\ (s, W_s) \epsilon S_h.$$

From Lemma 3.2, we get $(s_i, w_i) \epsilon S_h$, $i = 1, 2$ such that $\Phi(s_i, w_i) = t$, $s_1 < 0$ and $s_2 > 0$. This is, (s_i, w_i) $i = 1, 2$ is the solutions of (2.5) in S_h. Hence $s_1\varphi + w_1$, $s_2\varphi + w_2$ are two solutions of $(\mathbf{P}_t)$.

Proof of Theorem 2 Apply Lemma 3.2 and the proof of [4, Theorem 5.2].

Proof of Theorem 3 Apply Lemma 3.3 and the proof of [3, Proposition 2.4].

Remark However, we should point out that our method cannot be used to extend the results such as [4,prop.5.1] and [4, prop.6.1]].The reason is that the set S_h there is a smooth curve. In our method, we require S_h to be a connected set for which we can't use the concept of derivative.

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几类半线性椭圆共振问题

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摘要　设$\Omega\subset R^n$是一个有界正则区域，$\{\lambda_k\}$是$-\Delta$在$H_0(\Omega)$上的一列特征值．假定对某个给定的k，λ_k是单重的，φ为其相应的特征函数，$\int\varphi^2=1$．固定$h\in H^{-1}$使$\int h\varphi=0$．对于方程

$$(P_t)\qquad \begin{cases} -\Delta u-\lambda_k u+g(x,u)=t\varphi+h, \\ u=0. \qquad \partial\Omega \end{cases}$$

本文利用连通技巧和闭联集理论，推广了文［1］、［3］、［4］中的一些结果。我们获得

定理 1　假设$g:R\to R$满足

(g_1) g是具有周期原函数的连续周期函数，

$\lambda_k(k>1)$简单．如果对$\forall s\in R$，有

$$(H_4')\quad \begin{cases} \lambda_{k-1}<\lambda_k+g'(s)<\lambda_{k+1} \quad k>1. \\ \text{const}<\lambda_1+g'(s)<\lambda_2. \end{cases}$$

则$\forall h\in H^{-1}$，$\exists\tau_1,\tau_2\in R$．$\tau_1<0<\tau_2$使

(i)　(P_t)有解当且仅当$t\in[\tau_1,\tau_2]$．

(ii) 如果$t\in(\tau_1,\tau_2)-\{0\}$，则$(P_t)$至少有两个不同的解．

定理 2　假设(H_4')成立，λ_k简单，g满足

(H_2)　$\forall s,g$按x在Ω上可测；$g\in C^1$对a.e. $x\in\Omega$．

(H_5)　g有界

$$\lim_{|s|\to\infty} sg(x,s)=\mu>0.$$

则$\forall h\in H_0'$，$\exists\tau_1,\tau_2\in R$，$\tau_1<0<\tau_2$使

(i)　(P_t)有解当且仅当$t\in[\tau_1,\tau_2]$．

(ii) 若$t\in(\tau_1,\tau_2)-\{0\}$，则$(P_t)$至少有两个不同的解．

定理 3　［3, prop. 2.4］中的条件

$$q<\nu(-\Delta-\lambda_k I) \text{ 换成 } q\leq\nu(-\Delta-\lambda_k I)$$

结论仍然成立．